Math 1553 Exam 3 SOLUTIONS, Spring 2026, Ver. B

Name		**GT ID**	

Circle your instructor and lecture below. If your name or GT ID is not legible, or if you do not circle your lecture, we may deduct points.

Jankowski (A+HP, 8:00 AM) Jankowski (C, 9:00 AM) Callis (E, 10:00 AM)

Hao (F, 11:00 AM) Poudel (L, 4:00 PM) Van Why (M, 5:00 PM)

Poudel (S, 5:00 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- Calculators and cell phones are not allowed. Aids of any kind (notes, text, etc.) are not allowed. If you use pen, you must use black ink. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam.* If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, April 15.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

(a) If A is a 4×4 matrix and its columns are linearly independent, then $\det(A) = 0$.

○ True

● False

(b) An $n \times n$ matrix A is invertible if and only if $\lambda = 0$ is not an eigenvalue of A.

● True

○ False

(c) Let A be the 2×2 matrix that rotates vectors in $\mathbf{R}^2$ by $30°$ counterclockwise. Then A has exactly two different complex eigenvalues.

● True

○ False

(d) If A is a positive-stochastic $n \times n$ matrix, then $\mathrm{Nul}(A - I)$ must be a line.

● True

○ False

(e) If A is an invertible $n \times n$ matrix, then its eigenvalues are its diagonal entries.

○ True

● False

Problem 1 Solution.

(a) False: any square matrix A that has linearly independent columns is invertible, therefore $\det(A) \neq 0$.

(b) True: A is invertible if and only if $Ax = 0$ has only the trivial solution, which means that $Ax = 0x$ has only the trivial solution, so $\lambda = 0$ is not an eigenvalue of A.

Alternatively, we could recall the general fact that λ is an eigenvalue of A if and only if $A - \lambda I$ is not invertible, which means that $A - \lambda I$ is invertible if and only if λ is NOT an eigenvalue of A.

(c) True: if x is a nonzero vector in $\mathbf{R}^2$ then x and Ax are on different lines through the origin (since Ax is a 30 degree cc rotation of x), so A has no real eigenvalues. Therefore, the 2×2 real matrix A must have two complex eigenvalues that occur as a conjugate pair.

(d) True: it is a fact from section 5.6 that if A is positive-stochastic, then its 1-eigenspace is a line. Its 1-eigenspace is (by definition) $\mathrm{Nul}(A - I)$.

(e) False: for example, $A = \begin{pmatrix} 1 & 2 \\ -1 & 4 \end{pmatrix}$ has 1 as 4 as its diagonals, but its eigenvalues are $\lambda = 2$ and $\lambda = 3$.

2. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (4 points) Let A be an invertible 4×4 matrix. Which of the following statements **must** be true? Fill in the bubble for all that apply.

● If we do a row replacement on A to obtain a matrix B, then $\det(A) = \det(B)$.

○ $\det(-A) = -\det(A)$.

● If we swap the first and last rows of A to obtain a new matrix B, then $\det(A) = -\det(B)$.

● $\det(AA^{-1}) = 1$.

(b) (2 points) Find the value of a so that

$$\det\begin{pmatrix} 1 & 2 & 3 \\ -3 & 1 & 2 \\ 7 & 1 & 10 \end{pmatrix} = a \cdot \det\begin{pmatrix} 1 & 2 & 3 \\ -24 & 8 & 16 \\ 7 & 1 & 10 \end{pmatrix}.$$

Fill in the bubble for your answer below.

○ -1 ○ $\frac{1}{2}$ ○ $\frac{1}{4}$ ● $\frac{1}{8}$ ○ $-\frac{1}{8}$

○ 1 ○ 2 ○ 4 ○ 8 ○ none of these

(c) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} a-2d & b-2e & c-2f \\ a & b & c \\ 4g & 4h & 4i \end{pmatrix}$.

○ 1 ○ 2 ○ 3 ○ 4 ○ 6 ● 8

○ -1 ○ -2 ○ -3 ○ -4 ○ -6 ○ -8

○ 24 ○ -24 ○ none of these ○ not enough info

(d) (2 points) Find the area of the parallelogram with vertices

$$(1,2), \quad (4,5), \quad (5,-6), \quad (8,-3).$$

○ 3/2 ○ 3 ○ 7 ○ 7/2 ○ 6 ○ 15

○ 18 ○ 30 ● 36 ○ 40 ○ none of these

Problem 2 Solution.

(a) Statements (i) and (iii) are true, and they are standard facts about determinants.

Statement (ii) is not true: since A is 4×4, we know $\det(-A) = (-1)^4 \det(A) = \det(A)$.

Statement (iv) is true: $\det(AA^{-1}) = \det(I) = 1$.

(b) Call the first matrix A and the second matrix B. Note that we obtain B by multiplying the second row of A by 8.

This means $8\det(A) = \det(B)$, so $\det(A) = \frac{1}{8}\det(B)$ and therefore $a = \frac{1}{8}$.

(c) To get the final matrix from the first, we do the following row operations in order.

- $R_1 \leftrightarrow R_2$, multiplying the det by -1. The determinant is now -1.
- Multiply the new R_1 by -2, multiplying the new det by -2. The determinant is now $(-2)(-1) = 2$.
- Do $R_1 = R_1 + R_2$, which does not change the determinant. The det is still 2.
- Lastly, do $R_3 = 4R_3$, which multiplies the determinant by 4. This gives us the final answer of $4(2) = 8$.

(d) To find the vectors that determine the parallelogram, we find vectors that give two sides (starting from the initial point $(1, 2)$).

Vector from $(1, 2)$ to $(4, 5)$: $\begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

Vector from $(1, 2)$ to $(5, -6)$: $\begin{pmatrix} 4 \\ -8 \end{pmatrix}$.

The area of the parallelogram is therefore $\left|\det \begin{pmatrix} 3 & 4 \\ 3 & -8 \end{pmatrix}\right| = |-24 - 12| = 36$.

Another way to do this problem is to take the equivalent parallelogram with a vertex at the origin. To do this, we would shift $(1, 2)$ to the origin and shift the other points accordingly, which gives the parallelogram with vertices $(0, 0)$, $(3, 3)$, $(4, -8)$, and $(7, -5)$. Computing this area is identical to what we just did above.

3. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (3 pts) Let A be the 2×2 matrix that reflects each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = 9x$.

i. What are the eigenvalues of A? Fill in the bubble for your answer below.

○ 0 and 1 ○ 0 and -1 ● 1 and -1 ○ 9 and -1

ii. Which **one** of the following is an eigenvector of A?

● $\begin{pmatrix} -9 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 1/9 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

(b) (2 points) Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 6 & -2 & 0 \\ 2 & 2 & 0 \\ 1 & 2 & 2 \end{pmatrix}.$$

○ 0 and 1 ○ 1 and 2 ○ 1, 2, and 6 ○ -1, 2, and 6

● 2 and 4 ○ 2 and 6 ○ 2, $4 + 2\sqrt{2}$, and $4 - 2\sqrt{2}$

○ none of these

(c) (3 points) Let A be a 3×3 matrix. Which of the following statements must be true? Fill in the bubble for all that apply.

● If A has three different real eigenvalues, then it must be diagonalizable.

○ If v and w are linearly independent eigenvectors of A, then they must correspond to different eigenvalues.

● If A has $\lambda = 1$ as an eigenvalue with geometric multiplicity 3, then A is the 3×3 identity matrix.

(d) (2 points) Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}^{-1}$. Find $A^2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

○ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ ● $\begin{pmatrix} 27 \\ 9 \end{pmatrix}$ ○ $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 6 \\ 2 \end{pmatrix}$

○ $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 9 \\ 3 \end{pmatrix}$ ○ $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ ○ none of these

Problem 3 Solution.

(a) For part (i), note that A does not move any vector v along the line $y = 9x$ (in other words, $Av = v$).
Also, A flips every vector w on the line perpendicular to $y = 9x$, which is the line $y = -\frac{1}{9}x$. In other words, $Aw = -w$ for any w along the line $y = -\frac{1}{9}x$.

Therefore, the eigenvalues of A are 1 and -1.

For part (ii), there is only one nonzero vector that is in either of the eigenspaces described in part (i), and that is $\begin{pmatrix} -9 \\ 1 \end{pmatrix}$ which is in the (-1)-eigenspace.

(b) We expand $\det(A - \lambda I)$ along the third column.

$$\begin{aligned} 0 = \det(A - \lambda I) = \det \begin{pmatrix} 6-\lambda & -2 & 0 \\ 2 & 2-\lambda & 0 \\ 1 & 2 & 2-\lambda \end{pmatrix} \\ = (2-\lambda) * (-1)^{3+3} \det \begin{pmatrix} 6-\lambda & -2 \\ 2 & 2-\lambda \end{pmatrix} \\ = (2-\lambda)\Big[(6-\lambda)(2-\lambda)+4\Big] = (2-\lambda)(\lambda^2 - 8\lambda + 16) \\ = (2-\lambda)(\lambda-4)^2. \end{aligned}$$

Therefore, the eigenvalues are $\lambda = 2$ and $\lambda = 4$.

(c) Statement (i) is true. The three different real eigenvalues give three linearly independent eigenvectors, so A is diagonalizable by the Diagonalization Theorem.

Statement (ii) is not true. For example, e_1 and e_2 are linearly independent eigenvectors of I corresponding to the same eigenvalue $\lambda = 1$.

Statement (iii) is true. If $\lambda = 1$ has geometric multiplicity 3, then the 1-eigenspace of A is all of $\mathbf{R}^3$. This means that $Ax = x$ for every x in $\mathbf{R}^3$, so $A = I$.
Alternatively, we could use diagonalization. Since $\lambda = 1$ has geometric multiplicity 3 for our 3×3 matrix A, we know A is diagonalizable ($A = CDC^{-1}$ for some C) where the diagonal matrix is $D = I$, so $A = CIC^{-1} = CC^{-1} = I$.

(d) Note $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is the second column of C, so it is in the 3-eigenspace of A by the Diagonalization Theorem, thus $A^2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 3^2 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = 9 \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 27 \\ 9 \end{pmatrix}$.

4. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (d) are unrelated.

   (a) (4 points) Suppose A is a 4×4 matrix with characteristic polynomial $\det(A - \lambda I) = (1-\lambda)^2(2-\lambda)(6-\lambda)$. Which of the following statements must be true? Fill in the bubble for all that apply.

   - ● A is invertible.
   - ● The 6-eigenspace of A must be 1-dimensional.
   - ● If $\dim(\mathrm{Nul}(A - I)) = 2$, then A is diagonalizable.
   - ○ If v is an eigenvector in the 2-eigenspace of A, then $3v$ is an eigenvector in the 6-eigenspace of A.

   (b) (2 points) Suppose A is a 3×3 matrix. Which **one** of the following statements must be true?

   - ○ A must have either exactly one real eigenvalue or exactly three different real eigenvalues.
   - ○ If u and v are eigenvectors of A, then $u - v$ must also be an eigenvector of A.
   - ● If $\det(A) = 0$, then 0 must be an eigenvalue of A.
   - ○ At least one eigenvalue of A must have geometric multiplicity 2.

   (c) (2 points) Find the value of c so that $\lambda = 1$ is an eigenvalue of $\begin{pmatrix} c & 1 \\ -2 & 6 \end{pmatrix}$.

   ○ $c = -1/3$ ○ $c = 1/3$ ● $c = 3/5$ ○ $c = -3/5$

   ○ $c = -7/5$ ○ $c = 7/5$ ○ $c = -2$ ○ none of these

   (d) (2 points) In the space provided below, write an upper-triangular or lower triangular 2×2 matrix A that is **not invertible** and **not diagonalizable**. There is no partial credit, so write your answer carefully.

   $$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Problem 4 Solution.

(a) Statement (i) is true: $\det(A) = \det(A - 0I) = 1^2(2)(6) \neq 0$, so A is invertible.

Statement (ii) is true: we see $\lambda = 6$ has algebraic multiplicity 1, so it automatically has geometric multiplicity 1.

Statement (iii) is true: if the 1-eigenspace has dimension 2, then since the 2-eigenspace and 6-eigenspace are automatically each 1-dimensional, we get 4 linearly independent eigenvectors. Therefore, the 4×4 matrix A is diagonalizable.

Statement (iv) is not true: If a v is in the 2-eigenspace, then so is cv for any real number c, so here $3v$ is an eigenvector in the 2-eigenspace (not the 6-eigenspace) of A.

(b) Statements (i) and (ii) are false, for example consider $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. It has exactly two different real eigenvalues (not one or three). Also, the vectors $u = e_2$ and $v = e_3$ are eigenvectors of A, but $e_2 - e_3$ is not an eigenvector.

Statement (iii) is true: if $\det(A) = 0$ then A is not invertible, therefore $Ax = 0$ has infinitely many solutions and $\lambda = 0$ is an eigenvalue of A.

Statement (iv) is not true. For example, A might have three different real eigenvalues that each have geometric multiplicity 1.

(c) For $\lambda = 1$ to be an eigenvalue, we need $\det(A - 1I) = 0$.

$$0 = \det(A - 1I) = \det \begin{pmatrix} c-1 & 1 \\ -2 & 5 \end{pmatrix} = 5c - 5 - (1)(-2) = 5c - 3,$$

so $c = 3/5$.

(d) Many examples possible. Since the matrix is not invertible, we need $\lambda = 0$ to be an eigenvalue. Also, the matrix cannot have a second eigenvalue (in that case it would have two different real eigenvalues and be diagonalizable).

Therefore, we need $\lambda = 0$ to have algebraic multiplicity 2 but only geometric multiplicity 1. So our **triangular** A has 0 as its diagonal entries, and **any nonzero** entry as the non-diagonal entry (depending on whether upper or lower triangular).

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \text{ etc.} \quad \text{or} \quad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}, \text{ etc.}$$

5. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit.

For this problem, let $A = \begin{pmatrix} 3 & 5 & -15 \\ 0 & -2 & 15 \\ 0 & 0 & 3 \end{pmatrix}$.

(a) (2 points) Write the eigenvalues of A. You do not need to show your work on this part.
A is upper-triangular, so its eigenvalues are its diagonal entries: $\lambda = -2$ and $\lambda = 3$.

(b) (5 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.
For the 3-eigenspace:

$$\left(\, A - 3I \mid 0 \, \right) = \left(\begin{array}{ccc|c} 0 & 5 & -15 & 0 \\ 0 & -5 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives us x_1 free, $x_2 = 3x_3$, and x_3 free. We get parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ 3x_3 \\ x_3 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}. \qquad \text{Basis: } \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} \right\}$$

For the (-2)-eigenspace:

$$\left(\, A + 2I \mid 0 \, \right) = \left(\begin{array}{ccc|c} 5 & 5 & -15 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 5 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -3 & 0 \\ 0 & 0 & 15 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

This gives us $x_1 = -x_2$, x_2 free, and $x_3 = 0$. We get parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \\ 0 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad \text{Basis: } \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

(c) (3 pts) We form C using linearly independent eigenvectors and form D using the eigenvalues written **in the corresponding order**. Many answers are possible. For example,

$$C = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 1 \\ 0 & 1 & 0 \end{pmatrix}, D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad \text{or} \quad C = \begin{pmatrix} -1 & 1 & 0 \\ 1 & 0 & 3 \\ 0 & 0 & 1 \end{pmatrix}, D = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

6. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (4 points) Find $\det \begin{pmatrix} 1 & 1 & 3 & 4 \\ 4 & 0 & 0 & 0 \\ -7 & 3 & 4 & -3 \\ -8 & 0 & 1 & 2 \end{pmatrix}$. Enter your answer here: $\underline{-20}$.

Solution: Call the matrix A. We use the cofactor expansion along the 2nd row:

$$\begin{aligned} \det(A) &= 4C_{21} + 0 + 0 + 0 \\ &= 4(-1)^{2+1} \det \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & -3 \\ 0 & 1 & 2 \end{pmatrix} \\ &= -4\Big[1(4 \cdot 2 + 3) - 3(3 \cdot 2 - 0) + 4(3 \cdot 1 - 0)\Big] \\ &= -4\Big[11 - 18 + 12\Big] = -4(5) = -20. \end{aligned}$$

(b) (6 points) Find the *complex* eigenvalues of the matrix $A = \begin{pmatrix} 4 & -1 \\ 17 & 2 \end{pmatrix}$. For the eigenvalue with **positive** imaginary part, find a corresponding eigenvector v. Simplify your eigenvalues as much as possible! Enter your answers below. The eigenvalues are: $\lambda = 3 \pm 4i$.

The characteristic equation can be found the standard way or using the shortcut.

$$\begin{aligned} 0 = \det(A - \lambda I) &= \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) \\ &= \lambda^2 - (4+2)\lambda + (4(2) + 17) \\ &= \lambda^2 - 6\lambda + 25. \end{aligned}$$

This gives us

$$\lambda = \frac{6 \pm \sqrt{36 - 4(25)}}{2} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.$$

For an eigenvector for $\lambda = 3 + 4i$, we use the 2×2 eigenvector trick.

$$(A - (3+4i)I \mid 0) = \left(\begin{array}{cc|c} 4 - (3+4i) & -1 & 0 \\ (*) & (*) & 0 \end{array} \right) = \left(\begin{array}{cc|c} 1 - 4i & -1 & 0 \\ (*) & (*) & 0 \end{array} \right) = \left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array} \right)$$

so an eigenvector is $v = \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 - 4i \end{pmatrix}$.

Other answers are possible, for example

$$v = \begin{pmatrix} -1 \\ -1 + 4i \end{pmatrix}, \qquad v = \begin{pmatrix} 1 + 4i \\ 17 \end{pmatrix}, \qquad v = \begin{pmatrix} -1 - 4i \\ -17 \end{pmatrix},$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) Let $A = \begin{pmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{pmatrix}$.

i. (4 points) Find the steady-state vector w for A. Enter it in the space below.

Solution: We row-reduce $(A - I \mid 0)$:

$$(A - I \mid 0) = \left(\begin{array}{cc|c} -0.2 & 0.6 & 0 \\ 0.2 & -0.6 & 0 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & -3 & 0 \\ 0 & 0 & 0 \end{array} \right).$$

This gives us $x_1 = 3x_2$ and x_2 is free, so the 1-eigenspace is spanned by $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and the steady-state vector is $w = \frac{1}{3+1}\begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix}$.

ii. (2 points) What vector v does $A^n \begin{pmatrix} 60 \\ 100 \end{pmatrix}$ approach as n gets very large? Enter your answer in the space below. Briefly show your work.

Solution: $A^n \begin{pmatrix} 60 \\ 100 \end{pmatrix}$ approaches $(60 + 100)w = 160 \begin{pmatrix} 3/4 \\ 1/4 \end{pmatrix} = \begin{pmatrix} 120 \\ 40 \end{pmatrix}$.

(b) (4 points) Find a 2×2 matrix A whose (-1)-eigenspace is the span of $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ and whose 5-eigenspace is the span of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Write your answer in the space below.

Solution: We use the Diagonalization Theorem: $A = CDC^{-1}$ where C has eigenvectors as columns and D has the eigenvalues written in their corresponding order.

$$\begin{aligned} A &= \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -1 & 4 \end{pmatrix} \\ &= \begin{pmatrix} 4 & 3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 3 \\ -5 & 20 \end{pmatrix} \\ &= \begin{pmatrix} -19 & 72 \\ -6 & 23 \end{pmatrix} \end{aligned}$$

It is also possible to do the problem without diagonalization, by using linearity properties to compute Ae_1 and Ae_2 using the information given.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.