Math 1553 Exam 2 SOLUTIONS, Spring 2026, Ver. A

Name		**GT ID**	

Circle your instructor and lecture below. If your name or GT ID is not legible, or if you do not circle your lecture, we may deduct points.

Jankowski (A+HP, 8:00 AM)   Jankowski (C, 9:00 AM)   Callis (E, 10:00 AM)

Hao (F, 11:00 AM)   Poudel (L, 4:00 PM)   Van Why (M, 5:00 PM)

Poudel (S, 5:00 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- Calculators and cell phones are not allowed. Aids of any kind (notes, text, etc.) are not allowed. If you use pen, you must use black ink.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, March 11.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If $\{v_1, v_2, v_3\}$ is a linearly dependent set of vectors in $\mathbf{R}^5$, then the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ must have infinitely many solutions.

   ● True

   ○ False

   (b) Suppose V is a 4-dimensional subspace of $\mathbf{R}^n$, and let v_1, v_2, v_3, and v_4 be four linearly independent vectors in V. Then $\mathrm{Span}\{v_1, v_2, v_3, v_4\} = V$.

   ● True

   ○ False

   (c) If a homogeneous linear system consists of 4 equations in 8 variables, then its solution set is a subspace of $\mathbf{R}^8$.

   ● True

   ○ False

   (d) There is an invertible 3×3 matrix A with the property that

$$A \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = A \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}.$$

   ○ True

   ● False

   (e) If $T : \mathbf{R}^n \to \mathbf{R}^m$ is an onto linear transformation, then $n \leq m$.

   ○ True

   ● False

Problem 1 Solution.

(a) True, directly from the definition. The vectors are linearly dependent, so their corresponding homogeneous vector equation has more than just the trivial solution, therefore it has infinitely many solutions.

(b) True, directly from the Basis Theorem. Since V is 4-dimensional, any 4 linearly independent vectors in V will form a basis for V, so $\text{Span}\{v_1, v_2, v_3, v_4\} = V$.

(c) True: this is the null space of a 4×8 matrix, which is automatically a subspace of $\mathbf{R}^8$.

(d) False, by the Invertible Matrix Theorem. The transformation $T(x) = Ax$ is not one-to-one since $T(1, -1, 2) = T(0, 4, 3)$, so A is not invertible. Alternatively, we could note that, by linearity, $Ax = 0$ for the vector $x = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} - \begin{pmatrix} 0 \\ 4 \\ 3 \end{pmatrix}$. This means that the homogeneous matrix equation $Ax = 0$ has a non-trivial solution, so A has linearly dependent columns and is therefore not invertible.

(e) False. For example, take $n = 3$ and $m = 2$. Then $n > m$, but the transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (x, y)$ is onto.

2. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

   (a) (2 points) Let A be a 50×70 matrix with 30 pivots.

      i. What is the dimension of the column space of A?
      ○ 20 ● 30 ○ 40 ○ 50 ○ 70

      ii. What is the dimension of the null space of A?
      ○ 20 ○ 30 ● 40 ○ 50 ○ 70

      iii. Complete the sentence: Col(A) is a subspace of...
      ○ $\mathbf{R}^{20}$ ○ $\mathbf{R}^{30}$ ○ $\mathbf{R}^{40}$ ● $\mathbf{R}^{50}$ ○ $\mathbf{R}^{70}$

      iv. Complete the sentence: Nul(A) is a subspace of...
      ○ $\mathbf{R}^{20}$ ○ $\mathbf{R}^{30}$ ○ $\mathbf{R}^{40}$ ○ $\mathbf{R}^{50}$ ● $\mathbf{R}^{70}$

   (b) (3 points) Suppose that v_1, v_2, and v_3 are linearly independent vectors in $\mathbf{R}^n$. Which of the following must be true? Fill in the bubble for all that apply.

   ● If b is a vector and the vector equation $x_1v_1 + x_2v_2 = b$ is consistent, then it must have exactly one solution.

   ● The set $\{v_1, v_2, 5v_1 - 4v_2\}$ must be linearly dependent.

   ● The equation $x_1v_1 + x_2v_2 = v_3$ must be inconsistent.

   (c) (3 points) Let V be the set of vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ that satisfy $y \geq x^2$. Which of the subspace properties does V satisfy? Fill in the bubble for all that apply.

   ● V contains the zero vector.

   ○ V closed under addition. In other words, if u and v are vectors in V, then $u + v$ must be in V.

   ○ V closed under scalar multiplication. In other words, if u is a vector in V and c is a scalar, then cu must be in V.

   (d) (2 points) Suppose A is a 3×5 matrix and B is a 5×4 matrix. Which **one** of the following statements must be true?

   ○ If x is a vector and $Ax = 0$, then $ABx = 0$.

   ● If v is a vector in the column space of AB, then v must also be in the column space of A.

   ○ If v is a vector in the null space of AB, then v must also be in the null space of B.

   ○ If the homogeneous equation $Bx = 0$ has only the trivial solution, then the equation $ABx = 0$ has only the trivial solution.

Problem 2 Solution.

(a) By the Rank Theorem,

$$\dim(\text{Col A}) + \dim(\text{Nul A}) = \#(\text{columns of A}) = 70.$$

(i) We are told A has 30 pivots, so $\dim(\text{Col A}) = 30$.
(ii) The null space has dimension $70 - 30 = 40$.
(iii) The column space is a subspace of $\mathbf{R}^{\#\text{rows}}$, which is $\mathbf{R}^{50}$.
(iv) The null space is a subspace of $\mathbf{R}^{\#\text{columns}}$, which is $\mathbf{R}^{70}$.

(b) (i) is true: since $\{v_1, v_2, v_3\}$ is linearly independent, we know $\{v_1, v_2\}$ is linearly independent, so any consistent vector equation $x_1v_1 + x_2v_2 = b$ must have exactly one solution.

(ii) is true: $5v_1 - 4v_2$ is in $\text{Span}\{v_1, v_2\}$, so $\{v_1, v_2, 5v_1 - 4v_2\}$ must be linearly dependent, otherwise we would violate the Increasing Span Criterion.

(iii) is true: by the Increasing Span Criterion for the linearly independent set $\{v_1, v_2, v_3\}$, the vector v_3 cannot be in $\text{Span}\{v_1, v_2\}$, which is precisely to say that the equation $x_1v_1 + x_2v_2 = v_3$ is inconsistent.

(c) We can draw the region V without much trouble. It is the set of all points on and above the parabola $y = x^2$. From the picture, we can see that it contains the origin, however it will not be closed under addition (take vectors on its edge and add them, you will likely not be in V anymore) or scalar multiplication (any nonzero vector in V multiplied by -1 will no longer be in V).

(i) Yes, V contains the zero vector since $0 \geq 0^2 = 0$.

(ii) No, V is not closed under addition. For example, take $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$, both in V. However, $u + v = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$ which is not in V since $5 < 3^2$.

(iii) No, V is not closed under scalars. Take any nonzero vector in V, say $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then $-u = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$ which is not in V since $-1 < (-1)^2$.

(d) (i) is false: if $Ax = 0$, then x is in $\mathbf{R}^5$, so there is no such thing as "ABx". However, this statement still wouldn't be true even if A and B were $n \times n$ matrices.

(ii) is true, by the definition of multiplication. If v is in the column space of AB, then for some x we have $v = ABx = A(Bx)$, so v is "A times some vector" which is exactly what it means to be in the column space of A.

(iii) is false, take B to be any 5×4 matrix with a pivot in every column and take A to be the 3×5 zero matrix. If v is any nonzero vector in $\mathbf{R}^4$, then v is in $\mathrm{Nul}(AB)$ but is NOT in $\mathrm{Nul}(B)$.

(iv) is false, by the exact same argument as (iii).

3. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (3 points) Which of the following sets of vectors are linearly independent? Fill in the bubble for all that apply.

○ $\left\{ \begin{pmatrix} -3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ -4 \\ -2 \end{pmatrix} \right\}$

● $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$

● $\left\{ \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\}$

(b) (3 points) Which of the following are subspaces of $\mathbf{R}^3$? Fill in the bubble for all that apply.

● The range of the linear transformation $T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 5 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

○ The set of all vectors $\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathbf{R}^3$ satisfying $x_1 - x_2 + x_3 = 1$.

● Span $\left\{ \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(c) (2 pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ counterclockwise by $90°$, then reflects the vector across the line $y = x$.
Find $T\begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

○ $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ ● $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

(d) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be a linear transformation satisfying
$T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$ and $T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Find $T\begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

● $\begin{pmatrix} 17 \\ 3 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -7 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ ○ none of these

Problem 3 Solution.

(a) (i) is linearly dependent, since $v_2 = -2v_1$.
(ii) is linearly independent, since the corresponding matrix has a pivot in each column.
(iii) is linearly independent. In fact, we can do this without row-reduction by re-ordering the vectors so that the corresponding matrix is in REF with a pivot in each column. Recall that swapping columns in a matrix is invalid for solving linear systems, but for the purposes of testing linear independence, the order in which you write the vectors is irrelevant.

(b) (i) is a subspace of $\mathbf{R}^3$ because it is the column space of $\begin{pmatrix} 1 & -1 & 1 \\ 5 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}$.

(ii) is not a subspace, and in fact it fails the first property immediately because it does not contain the zero vector: $0 - 0 + 0 \neq 1$.

(iii) is a subspace, since the span of any vectors in $\mathbf{R}^3$ is automatically a subspace of $\mathbf{R}^3$.

(c) First we rotate $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$ counterclockwise 90 degrees to get $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, then we reflect that vector across the line $y = x$ to get $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Alternatively, we could have done this problem using matrices. The **rightmost** matrix is the one for the rotation since it is applied first, and the **leftmost** matrix is for the reflection.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

(d) This was nearly copied and pasted from the 3.3 Webwork. We could use the matrix A for T (so $T(x) = Ax$) by doing

$$A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \; T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix}, \qquad T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 17 \\ 3 \end{pmatrix}.$$

Alternatively, we could just use the properties of linearity:

$$T\begin{pmatrix} 3 \\ 2 \end{pmatrix} = T\begin{pmatrix} 3 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 2 \end{pmatrix} = 3T\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2T\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3\begin{pmatrix} 5 \\ 1 \end{pmatrix} + 2\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 17 \\ 3 \end{pmatrix}.$$

4. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (c) are unrelated.

(a) (4 points) Let $T : \mathbf{R}^c \to \mathbf{R}^d$ be a linear transformation with standard matrix A, so $T(x) = Ax$. Which of the following statements guarantee that T is one-to-one? Fill in the bubble for all that apply.

○ For each x in $\mathbf{R}^c$, there is at most one y in $\mathbf{R}^d$ so that $T(x) = y$.

● For each y in $\mathbf{R}^d$, there is at most one x in $\mathbf{R}^c$ so that $T(x) = y$.

○ $\text{Rank}(A) = d$.

● The equation $Ax = 0$ has only the trivial solution.

(b) (2 points) Suppose A is a matrix with columns v_1, v_2, and v_3, and suppose that the vector $\begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}$ is in the null space of A. In the box below, write a linear dependence relation for v_1, v_2, and v_3.

One linear dependence relation is $4v_1 + 2v_2 - 5v_3 = 0$

(c) (4 points) Suppose A is an $n \times n$ matrix. Which of the following conditions guarantee that A is invertible? Fill in the bubble for all that apply.

● $\text{Col}(A) = \mathbf{R}^n$.

● For each b in $\mathbf{R}^n$, the equation $Ax = b$ is consistent.

○ $Ax = b$ is consistent for some nonzero vector b.

● For some vector b in $\mathbf{R}^n$, the equation $Ax = b$ has exactly one solution.

Problem 4 Solution.

(a) Statement (i) does not guarantee one-to-one, and in fact it is satisfied by any transformation from $\mathbf{R}^c$ to $\mathbf{R}^d$.

Statement (ii) is nearly word-for-word the definition of one-to-one.

Statement (iii) does not guarantee T is one-to-one. For example, $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (x, y)$ is not one-to-one but its matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ satisfies $\text{rank}(A) = 2$.

Statement (iv) guarantees that T is one-to-one. If $Ax = 0$ has only the trivial solution, then A has a pivot in every column, so any matrix equation $Ax = b$ has at most one solution.

(b) Since $\begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}$ is the zero vector, we know

$$4v_1 + 2v_2 - 5v_3 = 0,$$

so this gives a linear dependence relation. Any nonzero scaling of the above equation is also a linear dependence relation, for example

$$8v_1 + 4v_2 - 10v_3 = 0, \qquad -4v_1 - 2v_2 + 5v_3 = 0.$$

(c) Statements (i) and (ii) each guarantee that A is invertible, and they come directly from the Invertible Matrix Theorem.

Statement (iii) does not guarantee that A is invertible.
For example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ then A is not invertible even though $Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is consistent.

Statement (iv) actually does guarantee that A is invertible. If $Ax = b$ has exactly one solution for some b, then $Ax = 0$ must also have exactly one solution (by a pivot argument or results from section 2.4), which means that A is invertible by the Invertible Matrix Theorem.

5. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.
For this page of the exam, consider the matrix A and its reduced row echelon form:

$$A = \begin{pmatrix} 1 & -7 & 3 & 10 \\ 3 & -21 & 4 & 15 \\ 2 & -14 & 0 & 2 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & -7 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Let T be the matrix transformation $T(x) = Ax$.

(a) (2 points) Write a basis for Col(A). There is no work required on this part.

(b) (4 points) Find a basis for Nul(A).

(c) (2 points) Write one vector x so that $T(x) = \begin{pmatrix} -7 \\ -21 \\ -14 \end{pmatrix}$. $x = \begin{pmatrix} \quad \end{pmatrix}$

There is no work required on this part and no partial credit.

(d) (2 pts) Are there two **different** vectors u and v (so $u \neq v$) satisfying $T(u) = T(v)$? If your answer is yes, write such vectors u and v. If your answer is no, justify why not.

Solution:

(a) The pivot columns of A will form a basis for Col(A): $\left\{ \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix} \right\}$. However, in this problem, any two columns of A will form a basis for Col(A) unless we choose the first two, which are scalar multiples of each other.

(b) From the RREF of A, for the solution set for $Ax = 0$ we see x_2 and x_4 are free and

$$x_1 - 7x_2 + x_4 = 0, \qquad x_3 + 3x_4 = 0.$$

Therefore, $x_1 = 7x_2 - x_4$, $x_3 = -3x_4$, and

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7x_2 - x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 7x_2 \\ x_2 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_4 \\ 0 \\ -3x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix}.$$

$$\text{Basis for Nul}(A): \left\{ \begin{pmatrix} 7 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

(c) The second column of A is $T(e_2)$, which the vector $\begin{pmatrix}-7\\-21\\-14\end{pmatrix}$. Therefore, $x = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ is a solution.

Alternatively, we could solve and find that the general solution to $Ax = \begin{pmatrix}-7\\-21\\-14\end{pmatrix}$ is $\begin{pmatrix}-7\\0\\0\\0\end{pmatrix} + x_2\begin{pmatrix}7\\1\\0\\0\end{pmatrix} + x_4\begin{pmatrix}-1\\0\\-3\\1\end{pmatrix}$, so any vector of that form is a solution. In particular, the vector $\begin{pmatrix}-7\\0\\0\\0\end{pmatrix}$ is one of the solutions.

(d) Many answers possible. One easy way to do this is to choose two different vectors in the nullspace. For example, $u = \begin{pmatrix}7\\1\\0\\0\end{pmatrix}$ and $v = \begin{pmatrix}-1\\0\\-3\\1\end{pmatrix}$.

Another example of a correct answer is $u = \begin{pmatrix}7\\1\\0\\0\end{pmatrix}$ and $v = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$.

6. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit.
Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation $T(x_1, x_2) = (x_1 - 4x_2,\ 2x_1 + 3x_2,\ 5x_1)$, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by $90°$ clockwise.

(a) Write the standard matrix A for T. Show your work.

$$A = \begin{pmatrix} T(e_1) & T(e_2) \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 5 & 0 \end{pmatrix}.$$

(b) Write the standard matrix B for U. Evaluate any trigonometric functions you write. Do not leave your answer in terms of sine and cosine.

$$B = \begin{pmatrix} \cos(90°) & \sin(90°) \\ -\sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(c) Which composition makes sense: $T \circ U$ or $U \circ T$? Fill in the correct bubble below. You do not need to show your work on this part.

● $T \circ U$ ○ $U \circ T$

(d) Compute the standard matrix C for the composition you selected in (c). Put your answer in the space provided below.
Solution:

$$C = AB = \begin{pmatrix} 1 & -4 \\ 2 & 3 \\ 5 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ -3 & 2 \\ 0 & 5 \end{pmatrix}.$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work will receive little or no credit. Parts (a) through (c) are unrelated.

   (a) This part is worth 4 points. Part (i) is 1 point, while part (ii) is 3 points.

      i. Which **one** of these must be equal to $(AB)^{-1}$, if A and B are invertible $n \times n$ matrices? There is no work necessary and no partial credit on this part.

      ○ $A^{-1}B^{-1}$ ● $B^{-1}A^{-1}$

      ii. Let $A = \begin{pmatrix} 3 & 2 \\ -1 & 7 \end{pmatrix}$. Find A^{-1}. Show your work, and enter your answer in the space provided below.

      Solution: Here $A = \begin{pmatrix} 3 & 2 \\ -1 & 7 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and since $ad - bc = 3(7) - 2(-1) = 23 \neq 0$, the matrix is invertible and

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{23}\begin{pmatrix} 7 & -2 \\ 1 & 3 \end{pmatrix}.$$

   (b) (4 points) Let $A = \begin{pmatrix} x & -3 \\ y & 5 \end{pmatrix}$.

   Find all values of x and y (if there are any) so that $A^2 = A$.
   Enter your answer in the box below.

$$x = -4 \qquad \text{and} \qquad y = \frac{20}{3}.$$

   Solution: We solve $A^2 = A$.

$$\begin{pmatrix} x & -3 \\ y & 5 \end{pmatrix}\begin{pmatrix} x & -3 \\ y & 5 \end{pmatrix} = \begin{pmatrix} x & -3 \\ y & 5 \end{pmatrix},$$

$$\text{so} \quad \begin{pmatrix} x^2 - 3y & -3x - 15 \\ xy + 5y & -3y + 25 \end{pmatrix} = \begin{pmatrix} x & -3 \\ y & 5 \end{pmatrix}.$$

   - We get $-3x - 15 = -3$, so $-3x = 12$ and $x = -4$.
   - We get $-3y + 25 = 5$, so $-3y = -20$ and $y = 20/3$.

   We could also check that $x^2 - 3y = x$ since $16 - 20 = -4$, and also check that $xy + 5y = y$ since

$$-4\left(\frac{20}{3}\right) + 5\left(\frac{20}{3}\right) = \frac{20}{3}.$$

(c) (2 pts) Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & 1 \end{pmatrix}$. Find all values of c (if there are any) so that $\begin{pmatrix} 2 \\ 1 \\ c \end{pmatrix}$ is in the null space of A. Enter your answer in the box below.

Solution: We compute $A \begin{pmatrix} 2 \\ 1 \\ c \end{pmatrix}$ and set it equal to the zero vector.

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ c \end{pmatrix} \begin{pmatrix} 0 \\ 2-2 \\ 4+1+c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 5+c \end{pmatrix}.$$

This is the zero vector when $5 + c = 0$, so $c = -5$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.