Studypalooza, Spring 2026

There are 50 problems for the Studypalooza event for Reading Day in Spring 2026. They were compiled using PreTeXt with the specific intention of producing a PDF that meets the WCAG 2.0 accessibility guidelines.

1. True or false: If $\{u, v, w\}$ is a set of linearly dependent vectors, then w must be a linear combination of u and v.

Solution 1.

False. For example, in $\mathbf{R}^2$ take $u = v = \begin{pmatrix}1\\0\end{pmatrix}$ and $w = \begin{pmatrix}0\\1\end{pmatrix}$.

2. Find the value of k that makes the following vectors linearly dependent: $\left\{\begin{pmatrix}-3\\0\\3\end{pmatrix}, \begin{pmatrix}3\\-3\\k\end{pmatrix}, \begin{pmatrix}3\\-1\\-1\end{pmatrix}\right\}$.

Solution 2.

Take the matrix A whose columns are the vectors and row-reduce a bit:

$$A = \begin{pmatrix}-3 & 3 & 3\\0 & -3 & -1\\3 & k & -1\end{pmatrix} \to \begin{pmatrix}-3 & 3 & 3\\0 & -3 & -1\\0 & k+3 & 2\end{pmatrix}$$

This matrix will have three pivots unless the second and third rows are multiples of each other. This means the third row is -2 times the second, so $-3(-2) = k + 3$, hence $k = 3$. Alternatively, the student could have computed $\det(A) = 9 - 3k$, so $k = 3$.

3. True or false: If $\{u, v\}$ is a basis for a subspace W, then $\{u - v, u + v\}$ is also a basis for W.

Solution 3.

True. By the Basis Theorem, any two linearly independent vectors in V will form a basis for V, so we just need to show $\{u-v, u+v\}$ is linearly independent. For this, suppose

$$x_1(u-v)+x_2(u+v)=0.$$

We expand and find

$$(x_1+x_2)u+(x_2-x_1)v=0.$$

Since u and v are linearly independent, this requires $x_1+x_2=0$ and $x_2-x_1=0$, which is only satisfied when $x_1=x_2=0$.

This shows that the vector equation $x_1(u-v)+x_2(u+v)=0$ has only the trivial solution $x_1=x_2=0$, so $\{u-v, u+v\}$ is linearly independent.

4. Which of the following are subspaces of $\mathbf{R}^4$?

(a) The set $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; 2x-y-z=0 \right\}$

(b) The set of solutions to $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & -1 \end{pmatrix} x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Solution 4.

(a) Yes, W is a subspace of $\mathbf{R}^4$ since it is $\mathrm{Nul}\begin{pmatrix} 2 & -1 & -1 & 0 \end{pmatrix}$.

(b) No, it is not a subspace. In fact, it fails the very first property of subspaces since it does not contain $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

5. True or false: The set W of vectors $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ with $abc=0$. Then W is closed under addition.

Solution 5.

False. For example, the vectors $u = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ are in W, but $u + v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ which is not in W since $1 \cdot 1 \cdot 1 \neq 0$.

6. Write the matrices for each of the following.

(a) Counterclockwise rotation by 90°.

(b) Reflection across the line $y = x$.

(c) Clockwise rotation by 90°

(d) Reflection across the x-axis.

(e) Reflection across the y-axis.

Solution 6.

(a) Counterclockwise rotation by 90°: $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(b) Reflection across the line $y = x$: $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(c) Clockwise rotation by 90°: $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

(d) Reflection across the x-axis: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

(e) Reflection across the y-axis: $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$.

7. Find all k so that $\begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix}$ and $\begin{pmatrix} k \\ 1 \\ -6 \end{pmatrix}$ are orthogonal.

Solution 7.

We set $0 = \begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix} \cdot \begin{pmatrix} k \\ 1 \\ -6 \end{pmatrix} = 2k + k - 6$, so $k = 2$.

8. Find all k so that the matrix transformation corresponding to the following matrix is not onto: $\begin{pmatrix} 1 & 3 & 9 \\ 2 & 6 & k \end{pmatrix}$.

Solution 8.

The columns of the matrix will span \$$\mathbf{R}^2$\$ unless they are all collinear, in which case $k = 18$. Alternatively, we could find k by row-reducing and determining when the matrix will fail to have a pivot in the second row.

9. Let $T : \mathbf{R}^a \to \mathbf{R}^b$ be a transformation. Match the four statements with their corresponding terms listed below.

(a) For each y in $\mathbf{R}^b$, there is at most one x in $\mathbf{R}^a$ so that $T(x) = y$.
(b) For each y in $\mathbf{R}^b$, there is at least one x in $\mathbf{R}^a$ so that $T(x) = y$.
(c) For each y in $\mathbf{R}^b$, there is at exactly one x in $\mathbf{R}^a$ so that $T(x) = y$.
(d) For each x in $\mathbf{R}^a$, there is exactly one y in $\mathbf{R}^b$ so that $T(x) = y$.

The terms are: "transformation," "one-to-one," "onto," and "invertible."

Solution 9.

(a) One-to-one, directly from the definition. This says that any y in the codomain of T is the image of at most one x in the domain.
(b) Onto, directly from the definition.
(c) Invertible: this statement says T is one-to-one and onto.
(d) Transformation: directly from the definition.

10. True or false: Suppose A is a 4×6 matrix. Then the dimension of the null space of A is at most 2.

Solution 10.

False. Just take A to be the 4×6 zero matrix, then its null space has dimension 6.

11. Complete the entries of A so that $\mathrm{Col}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$ and $\mathrm{Nul}(A) = \mathrm{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$.

$A = \begin{pmatrix} r & 1 \\ s & 2 \end{pmatrix}$.

Solution 11.

For $\mathrm{Col}(A)$ to be the desired line, $\begin{pmatrix}r\\s\end{pmatrix}$ must be a scalar multiple of $\begin{pmatrix}1\\2\end{pmatrix}$.

We also need $\begin{pmatrix}1\\1\end{pmatrix}$ to be in the null space of A, and solving $A\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$ gives us $r + 1 = 0$ and $s + 2 = 0$, so $r = -1$ and $s = -2$.

12. Suppose $T : \mathbf{R}^7 \to \mathbf{R}^9$ is a linear transformation with standard matrix A, and suppose the range of T has a basis consisting of 3 vectors. What is the nullity of A?

Solution 12.

We are given that the dimension of the range of T is 3, which means that the rank of A is 3. From the codomain and domain of T, we know that A is a 9×7 matrix. By the Rank Theorem: $\mathrm{nullity}(A) = 7 - \mathrm{rank}(A) = 7 - 3 = 4$.

13. Define $T : \mathbf{R}^3 \to \mathbf{R}^4$ by $T(x, y, z) = (0, x - y, y - x, z)$.

Is T one-to-one? Is T onto?

Solution 13.

No, T is not one-to-one. One way to see this is that the equation $T(v) = 0$ has infinitely many solutions, since for any real number c we get $T(c, c, 0) = (0, c - c, c - c, 0) = (0, 0, 0, 0)$.

Also, T is not onto: every vector in the range of T has 0 as its first entry, so for example $(1, 0, 0, 0)$ is not in the range of T.

14. Suppose A is a 7×5 matrix and its null space is a line, and let T be the matrix transformation $T(x) = Ax$. What is true about the range of T?

(a) It is a 4-dimensional subspace of $\mathbf{R}^5$.

(b) It is a 6-dimensional subspace of $\mathbf{R}^7$.

(c) It is a 4-dimensional subspace of $\mathbf{R}^7$.

(d) It is a 6-dimensional subspace of $\mathbf{R}^5$.

(e) none of these

Solution 14.

Since A is 7×5, we know its column space is a subspace of $\mathbf{R}^7$. From the fact that the null space of A is a line, we have $\operatorname{rank}(A) = 5 - 1 = 4$ by the Rank Theorem, so the column space of A is 4-dimensional.

Since the range of T is the column space of A, this means that the range of T is a 4-dimensional subspace of $\mathbf{R}^7$.

15. Say that $S : \mathbf{R}^2 \to \mathbf{R}^3$ and $T : \mathbf{R}^3 \to \mathbf{R}^4$ are linear transformations. Which one of the following must be true about $T \circ S$?

(a) It is one-to-one

(b) It is not one-to-one

(c) It is onto

(d) The composition $T \circ S$ is not defined

(e) It is not onto

Solution 15.

The transformation $T \circ S$ might be one-to-one. For example, if $S(x, y) = (x, y, 0)$ and $T(x, y, z) = (x, y, z, 0)$, then $(T \circ S)(x, y) = (x, y, 0, 0)$, so this particular $T \circ S$ is one-to-one.

However, $T \circ S$ cannot be onto because it corresponds to multiplication by a 4×2 matrix, which has a max of 2 pivots and therefore cannot have a pivot in every row.

16. True or false: Suppose that A is an invertible $n \times n$ matrix. Then $A + A$ must be invertible.

Solution 16.

True, since $(2A)^{-1} = \frac{1}{2}A^{-1}$. To confirm this, we can compute $(2A)\left(\frac{1}{2}A^{-1}\right) = \left(2 \cdot \frac{1}{2}\right)AA^{-1} = I$, and similarly $\left(\frac{1}{2}A^{-1}\right)(2A) = I$.

17. True or false: Suppose A is a 3×3 matrix and the equation $Ax = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ has exactly one solution. Then A must be invertible.

Solution 17.

True. Since $Ax = \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix}$ has exactly one solution, we conclude A has a pivot in every column, so A is invertible by the Invertible Matrix Theorem.

18. Suppose A and B are $n \times n$ matrices and AB is not invertible. Which one of the following must be true?

(a) A is not invertible.

(b) B is not invertible.

(c) At least one of the matrices A and B is not invertible.

(d) none of these.

Solution 18.

At least one of A and B is not invertible. One way to see this is through the formula for the determinant. Since AB is not invertible, we know $0 = \det(AB) = \det(A)\det(B)$, so it must be true that $\det(A) = 0$ or $\det(B) = 0$ (or both).

We cannot conclude that A is the one that must not be invertible, or that B is the one that must not be invertible. For example, it could be the case that $A = I$ and $B = 0$, or that $A = 0$ and $B = I$.

19. Suppose A and B are 3×3 matrices, with $\det(A) = 3$ and $\det(B) = -6$. Find $\det(2A^{-1}B)$.

Solution 19.

Multiplying a 3×3 matrix by 2 will multiply each row of the matrix by 2, so it will multiply the determinant by 8.

Using this fact and the properties of the determinant, we compute $\det(2A^{-1}B) = 8\det(A^{-1}B) = 8\det(A^{-1})\det(B) = 8 \cdot \dfrac{1}{3} \cdot (-6) = -16.$

20. Let A be the 3×3 matrix satisfying $Ae_1 = e_3$, $Ae_2 = e_2$, and $Ae_3 = 2e_1$ (recall that e_1, e_2, and e_3 are the standard basis vectors of $\mathbf{R}^3$).

Find $\det(A)$.

Solution 20.

$A = (\,Ae_1 \quad Ae_2 \quad Ae_3\,)$, so $\det(A) = \det\begin{pmatrix} 0 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = -2.$

21. Suppose A is a square matrix and $\lambda = -1$ is an eigenvalue of A. Which one of the following must be true?

(a) $\mathrm{Nul}(A+I) = \{0\}$.

(b) A is invertible.

(c) The columns of A are linearly independent.

(d) For some nonzero vector x, the vectors Ax and x are linearly dependent.

(e) The equation $Ax = x$ has only the trivial solution.

Solution 21.

(a) Not true, in fact $\mathrm{Nul}(A+I)$ must have dimension at least 1 since $\lambda = -1$ is an eigenvalue of A.

(b) Not necessarily true. For example, $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible even though it has $\lambda = -1$ as an eigenvalue.

(c) Not necessarily true. For example, if $A = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$, then $\lambda = -1$ is an eigenvalue of A, but the columns of A are not linearly independent.

(d) True: for any nonzero vector in the (-1)-eigenspace of A, we have $Ax = -x$, so $\{Ax, x\}$ is a linearly dependent set.

(e) Not necessarily true. For example, if $A = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ then $\lambda = -1$ is an eigenvalue of A, but the equation $Ax = x$ has infinitely many solutions since $\lambda = 1$ is an eigenvalue of A.

22. Suppose A is a 4×4 matrix with characteristic polynomial $-\lambda(1-\lambda)^2(5-\lambda)$.

What is the rank of A?

Solution 22.

From the characteristic polynomial of A, we see that $\lambda = 0$ is an eigenvalue of A with algebraic multiplicity 1, so its geometric multiplicity must also be 1. In other words, the 0-eigenspace is 1-dimensional.

Therefore, $\text{rank}(A) = 4 - 1 = 3$.

23. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects across the line $x_2 = 2x_1$. Find the value of k so that $T\begin{pmatrix}2\\k\end{pmatrix} = \begin{pmatrix}2\\k\end{pmatrix}$.

Solution 23.

Recall that for any 2×2 reflection matrix A, the line of reflection is the 1-eigenspace of A. In other words, $Ax = x$ for every x on the line of reflection. Here, the standard matrix for T is the matrix A that reflects vectors across $x_2 = 2x_1$.

Therefore, to solve $T\begin{pmatrix}2\\k\end{pmatrix} = \begin{pmatrix}2\\k\end{pmatrix}$, we need $\begin{pmatrix}2\\k\end{pmatrix}$ to be on the line $x_2 = 2x_1$. Thus $k = 2(2) = 4$.

24. Find the value of k so that $\begin{pmatrix}1 & k\\1 & 3\end{pmatrix}$ has one real eigenvalue with algebraic multiplicity 2.

Solution 24.

The characteristic polynomial of the matrix is
$(1-\lambda)(3-\lambda) - k = \lambda^2 - 4\lambda + 3 - k = (\lambda - 2)^2 - 1 - k$.

For this polynomial to have only one real root with algebraic multiplicity 2, it must be a perfect square, so $-1 - k = 0$, thus $k = -1$.

25. Suppose A is a diagonalizable matrix with characteristic polynomial
$\det(A - \lambda I) = (1-\lambda)^3(2-\lambda)(3-\lambda)$.

(a) What is the dimension of the 1-eigenspace of A?

(b) For some n, the 1-eigenspace of A is a subspace of $\mathbf{R}^n$. What is n?

Solution 25.

(a) Since A is diagonalizable and the eigenvalue $\lambda = 1$ has algebraic multiplicity 3, the dimension of the 1-eigenspace must also equal 3.

(b) The characteristic polynomial of A has degree 5, so the matrix A must be 5×5. This means the 1-eigenspace of A is a subspace of $\mathbf{R}^5$, so $n = 5$.

26. Find the value of t so that $\lambda = 3$ is an eigenvalue of $\begin{pmatrix} 1 & t & 3 \\ 1 & 1 & 1 \\ 3 & 0 & 3 \end{pmatrix}$.

Solution 26.

We use the cofactor expansion along the third row to solve:

$0 = \det(A - 3I) = \det \begin{pmatrix} -2 & t & 3 \\ 1 & -2 & 1 \\ 3 & 0 & 0 \end{pmatrix} = 3(t + 6)$, so $t = -6$.

27. Suppose A is a 2×2 matrix with characteristic polynomial $(1 - \lambda)(2 - \lambda)$. Find the characteristic polynomial of A^2.

Solution 27.

We observe that if λ is an eigenvalue for a square matrix A, then λ^2 is an eigenvalue for A^2, since for any λ-eigenvector v of A:

$A^2 v = A(Av) = A(\lambda v) = \lambda A v = \lambda(\lambda v) = \lambda^2 v.$

Here, the eigenvalues of A are $\lambda = 1$ and $\lambda = 2$, so the eigenvalues of A^2 are $1^2 = 1$ and $2^2 = 4$. These are the only possible eigenvalues of A since a 2×2 matrix has at most 2 eigenvalues. Therefore, the characteristic polynomial of A^2 is $(1 - \lambda)(4 - \lambda)$.

28. Suppose that x is an eigenvector of a matrix A corresponding to $\lambda = 3$ and that x is also an eigenvector of a matrix B corresponding to $\lambda = 4$.

Determine if x an eigenvector for $2A - B$. If so, find the corresponding eigenvalue.

Solution 28.

We compute $(2A - B)(x) = 2Ax - Bx = 2(3x) - 4x = 2x$. Therefore, x is an eigenvector of $2A - B$ corresponding to the eigenvalue $\lambda = 2$.

29. Suppose A is a 4×4 matrix with eigenvalues $0, 1$, and 2, where the eigenvalue 1 has algebraic multiplicity 2. Which of the following statements must be true? Select all that apply.

(a) A is not invertible.

(b) A is not diagonalizable.

Solution 29.

(a) True: A is not invertible, since it has $\lambda = 0$ as an eigenvalue.

(b) Not necessarily true. it is possible for A to be diagonalizable, for example

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

30. True or false: Suppose A is a 5×5 matrix with real entries. Then A must have at least one real eigenvalue.

Solution 30.

True. The characteristic polynomial of an $n \times n$ matrix A is a degree n polynomial whose coefficients are real numbers, and its roots (which occur in conjugate pairs) are the eigenvalues of A.

This guarantees A has at least one real eigenvalue if n is odd, which is the case here since $n = 5$.

31. Suppose A is a positive stochastic matrix satisfying $A\begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$, and let $v = \begin{pmatrix} 5 \\ 95 \end{pmatrix}$.

As n gets very large, what vector does $A^n v$ approach?

Solution 31.

We are implicitly told that the steady-state vector for A is $w = \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix}$, therefore $A^n v$ approaches cw as $n \to \infty$, where c is the sum of entries in v. Therefore,

$A^n v \to (5 + 95) \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} = 100 \begin{pmatrix} 3/5 \\ 2/5 \end{pmatrix} = \begin{pmatrix} 60 \\ 40 \end{pmatrix}$.

32. Suppose A is a 4×4 matrix with rank 2. Which one of the following must be true?

(a) A must have four distinct eigenvalues.
(b) A is not diagonalizable.
(c) A is diagonalizable.
(d) A cannot have four distinct eigenvalues.

Solution 32.

(a) False: for example, take $A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(b) False: for example, the matrix above is diagonalizable.

(c) False: for example, take $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then A has rank 2 and $\lambda = 0$ has algebraic multiplicity 3 but geometric multiplicity of only 2, so A is not diagonalizable.

(d) True. Since A is a 4×4 matrix with rank two, its 0-eigenspace (i.e. its null space) has dimension $4 - 2 = 2$. Therefore, the algebraic multiplicity of $\lambda = 0$ is at least 2. Since the sum of algebraic multiplicities is 4, this means A has at most 3 eigenvalues.

33. Suppose A is a 2×2 matrix whose entries are real numbers, and suppose that $\lambda = 1 + i$ is an eigenvalue of A with corresponding eigenvector $\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$. Which one of the following statements must be true?

(a) A must have eigenvalue $1 - i$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1+i \end{pmatrix}$.

(b) A must have eigenvalue $1 - i$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$.

(c) A must have eigenvalue $1 + i$ with corresponding eigenvector $\begin{pmatrix} 2 \\ 1-i \end{pmatrix}$.

(d) none of these

Solution 33.

The answer is (b). By the theory of complex eigenvalues, we know that if λ is an eigenvalue of a real matrix A with eigenvector v, then $\bar{\lambda}$ is also an eigenvalue of A with eigenvector $\bar{v}$ (where $\bar{v}$ means we take the complex conjugate of each entry of v).

In the context of this problem, this means that $\bar{\lambda} = 1 - i$ is an eigenvalue of A with corresponding eigenvector $\bar{v} = \begin{pmatrix} 2 \\ 1-i \end{pmatrix}$.

34. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation 45° clockwise, and let A be the standard matrix for T.

Which one of the following statements is true?

(a) A has two distinct real eigenvalues.

(b) A has one complex eigenvalue with algebraic multiplicity 2.

(c) A has one real eigenvalue with algebraic multiplicity 2.

(d) A has two distinct complex eigenvalues.

Solution 34.

The answer is (d). Let v be any nonzero vector in $\mathbf{R}^2$. We know from geometry that Av is not on the same line through the origin in $\mathbf{R}^2$ as v. In other words, Av is not a real scalar multiple of v.

This shows that A has no real eigenvalues, so its eigenvalues are complex. Since A is a 2×2 matrix

35. Suppose u and v are orthogonal unit vectors (recall that "unit" vector means length 1).

Find the dot product $(3u - 8v) \cdot 4u$.

Solution 35.

We are given $||u|| = ||v|| = 1$ and $u \cdot v = 0$.

We compute:

$$\begin{aligned}(3u - 8v) \cdot 4u &= (3u \cdot 4u) - (8v \cdot 4u) \\ &= 12||u||^2 - 32v \cdot u \\ &= 12(1) - 32(0) = 12\end{aligned}$$

36. Find all values of k so that $\begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix}$ and $\begin{pmatrix} k \\ 1 \\ -6 \end{pmatrix}$ are orthogonal.

Solution 36.

We solve: $0 = \begin{pmatrix} 2 \\ k \\ 1 \end{pmatrix} \cdot \begin{pmatrix} k \\ 1 \\ -6 \end{pmatrix} = 2k + k - 6 = 3k - 6$, so $k = 2$.

37. True or false: If W is a subspace of $\mathbf{R}^{100}$ and v is in $W^\perp$, then the orthogonal projection of v onto W must be the zero vector.

Solution 37.

True. This is a geometric fact, but we can also see it through the Orthogonal Decomposition Theorem: we are told that $v = v_{W^\perp}$, so the orthogonal decomposition of v with respect to W is $v = v_W + v_{W^\perp} = v_W + v$.

From the fact that $v = v_W + v$ we get $v_W = 0$. In other words, the orthogonal projection of v onto W is the zero vector.

38. True or false: Suppose W is a subspace of $\mathbf{R}^n$. If x is a vector in $\mathbf{R}^n$ and x_W is the orthogonal projection of x onto W, then $x \cdot x_W$ must be 0.

Solution 38.

False. If x is a nonzero vector in W, then $x = x_W$ so $x \cdot x_W = x \cdot x = ||x||^2 > 0$.

As a specific example, if W is the x_1-axis in $\mathbf{R}^2$ and $x = \begin{pmatrix}1\\0\end{pmatrix}$, then $x_W = \begin{pmatrix}1\\0\end{pmatrix}$ and $x \cdot x_W = 1$.

39. Suppose A is an invertible 3×3 matrix. What is the dot product of the second row of A and the third column of AA^{-1}?

- (a) 0
- (b) 1
- (c) -1
- (d) 2
- (e) -2
- (f) not enough info

Solution 39.

The answer is (a). Recall that $AA^{-1} = I$, so the given product is the entry in the second row and third column of I, which is 0.

40. Find the orthogonal projection of $\begin{pmatrix}0\\1\end{pmatrix}$ onto $\text{Span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}$.

Solution 40.

The matrix for projection is

$$\frac{1}{u\cdot u}uu^T = \frac{1}{1+4}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}.$$

Therefore, our answer is $\frac{1}{5}\begin{pmatrix}1 & 2\\2 & 4\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}2/5\\4/5\end{pmatrix}$.

41. Find the orthogonal projection of $\begin{pmatrix}6\\5\\4\end{pmatrix}$ onto $\operatorname{Span}\left\{\begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$.

Solution 41.

With $A = \begin{pmatrix}1 & 1\\0 & 1\\1 & 0\end{pmatrix}$, we solve $A^TAx = A^T\begin{pmatrix}6\\5\\4\end{pmatrix}$.

This gives $\begin{pmatrix}2 & 1\\1 & 2\end{pmatrix}x = \begin{pmatrix}10\\11\end{pmatrix}$, so $x = \begin{pmatrix}3\\4\end{pmatrix}$ and the projection is $Ax = \begin{pmatrix}7\\4\\3\end{pmatrix}$.

42. Let B be the standard matrix for the orthogonal projection of $\mathbf{R}^3$ onto $W = \left\{\begin{pmatrix}x\\y\\z\end{pmatrix} \text{ in } \mathbf{R}^3 \,\middle|\, x+y+2z=0\right\}$.

What is the dimension of the 1-eigenspace of B?

Solution 42.

Orthogonal projection onto a subspace fixes all vectors in that subspace, so $Bx = x$ for all x in W. Since $\dim(W) = 2$, the 1-eigenspace of B has dimension 2.

43. Let W be the subspace of $\mathbf{R}^4$ given by all vectors $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}$ satisfying $x + y + z + w = 0$. Find the dimension of $W^\perp$.

Solution 43.

Note that $W = \operatorname{Nul}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$, and the system $\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 0 \end{array}\right)$ has 3 free variables, so $\dim(W) = 3$.

Since W is a subspace of $\mathbf{R}^4$, it follows that $4 = \dim(W) + \dim(W^\perp) = 3 + \dim(W^\perp)$, so $\dim(W^\perp) = 1$.

44. True or false: If b is a vector in the column space of a matrix A, then every solution to $Ax = b$ is also a least-squares solution to $Ax = b$.

Solution 44.

True. The idea behind least squares for $Ax = b$ is to solve for x so that Ax is the closest vector to b in the column space of A. But if b is already in the column space of A, then that "closest" vector is just b itself, so we are just solving $Ax = b$.

We give an alternative explanation using purely algebraic tools. Since b is in $\operatorname{Col}(A)$, we have $b = b_{\operatorname{Col}(A)}$. Therefore, the equation $Ax = b$ is identical to the least-squares equation $Ax = b_{\operatorname{Col}(A)}$.

45. True or false: If A is an $m \times n$ matrix, b is in $\mathbf{R}^m$, and $\hat{x}$ is a least-squares solution to $Ax = b$, then $\hat{x}$ is the point in $\operatorname{Col}(A)$ that is closest to b.

Solution 45.

False: $A\hat{x}$ is the closest point to b in the column space of A.

46. Find the least-squares solution $\hat{x}$ to $\begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} x = \begin{pmatrix} 14 \\ -2 \\ 0 \end{pmatrix}$.

Solution 46.

Here $A = \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix}$ and $b = \begin{pmatrix} 14 \\ -2 \\ 0 \end{pmatrix}$. Solving $A^T A \hat{x} = A^T b$ gives

$$\begin{pmatrix} 6 & -2 & -2 \end{pmatrix} \begin{pmatrix} 6 \\ -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 6 & -2 & -2 \end{pmatrix} \begin{pmatrix} 14 \\ -2 \\ 0 \end{pmatrix}.$$

Computing both sides gives $44\hat{x} = 88$, so $\hat{x} = 2$.

47. Find the best fit line $y = Mx + B$ for the data points below using least squares: $(-7, -22)$, $(0, -2)$, and $(7, 6)$.

Solution 47.

Just by plotting the points, we can tell that there is clearly no line through all three of them, so the corresponding system of equations written below is inconsistent.

$$\begin{aligned} -22 &= -7M + B, \\ -2 &= 0M + B, \\ 6 &= 7M + B. \end{aligned}$$

This is the system $Ax = b$ given by $\begin{pmatrix} -7 & 1 \\ 0 & 1 \\ 7 & 1 \end{pmatrix} x = \begin{pmatrix} -22 \\ -2 \\ 6 \end{pmatrix}$.

Solving $A^T A \hat{x} = A^T b$, we get $A^T A = \begin{pmatrix} 98 & 0 \\ 0 & 3 \end{pmatrix}$ and $A^T \begin{pmatrix} -22 \\ -2 \\ 6 \end{pmatrix} = \begin{pmatrix} 196 \\ -18 \end{pmatrix}$. Solving $\left(\begin{array}{cc|c} 98 & 0 & 196 \\ 0 & 3 & -18 \end{array}\right)$ gives $\hat{x} = \begin{pmatrix} 2 \\ -6 \end{pmatrix}$, so the line is $y = 2x - 6$.

48. Let $A = \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix} \begin{pmatrix} -3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ 5 & 2 \end{pmatrix}^{-1}$. Find $A^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Solution 48.

The matrix A has been diagonalized for us.

Note that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ spans the (-1)-eigenspace of A, so $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

Repeated multiplication by A just flips $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ back and forth between itself and its negative, so we get $A^2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $A^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -2 \end{pmatrix}$.

49. True or false: If A is a diagonalizable 6×6 matrix, then A has 6 different eigenvalues.

Solution 49.

False. Just take A to be the 6×6 identity matrix. Then A is diagonalizable (in fact, diagonal!) but it only has one eigenvalue, namely $\lambda = 1$.

50. Find the eigenvalues of $A = \begin{pmatrix} 1 & 4 \\ 4 & 7 \end{pmatrix}$.

Solution 50.

We solve: $0 = \det(A - \lambda I) = (1-\lambda)(7-\lambda) - 16 = \lambda^2 - 8\lambda - 9 = (\lambda+1)(\lambda-9)$.

Therefore, the eigenvalues of A are $\lambda = -1$ and $\lambda = 9$.