**Math 1553 Worksheet §2.5, 2.6, 2.7, 2.9, 3.1**

Solutions

**1.** If the statement is always true, circle TRUE. Otherwise, circle FALSE. Justify your answer.

**a)** Suppose $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $A\begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Must $v_1, v_2, v_3$ be linearly dependent? If true, write a linear dependence relation for the vectors.
**TRUE** **FALSE**

**b)** In the following, $A$ is an $m \times n$ matrix.

(1) **TRUE** **FALSE** If $A$ has linearly independent columns, then $Ax = b$ must have at least one solution for each $b$ in $\mathbf{R}^m$.

(2) **TRUE** **FALSE** If $b$ is a vector in $\mathbf{R}^m$ and $Ax = b$ has exactly one solution, then $m \geq n$.

## Solution.

**a)** **TRUE**. By definition of matrix multiplication, $-3v_1+2v_2+7v_3 = 0$, so $\{v_1, v_2, v_3\}$ is linearly dependent and the equation gives a linear dependence relation.

**b)** (1) **FALSE** For example $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. There is no solution for $Ax = b$. (Note, however: if $A$ has linearly independent columns, then the system $Ax = 0$ has no free variables, so $Ax = b$ is either inconsistent or has a unique solution.)

(2) **TRUE** If $Ax = b$ has a unique solution, then since it is a translation of the solution set to $Ax = 0$, this means that $Ax = 0$ has only the trivial solution (no free variables). Thus, $A$ has a pivot in every column, which is impossible if $m < n$ (i.e. impossible if $A$ has more columns than rows), so $m \geq n$.

**2.** Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

**a)** If $A$ is a $3 \times 10$ matrix with 2 pivots, then $\dim(\text{Nul}A) = 8$ and $\text{rank}(A) = 2$.

**TRUE** **FALSE**

**b)** If $\{a, b, c\}$ is a basis of a subspace $V$, then $\{a, a + b, b + c\}$ is a basis of $V$ as well.

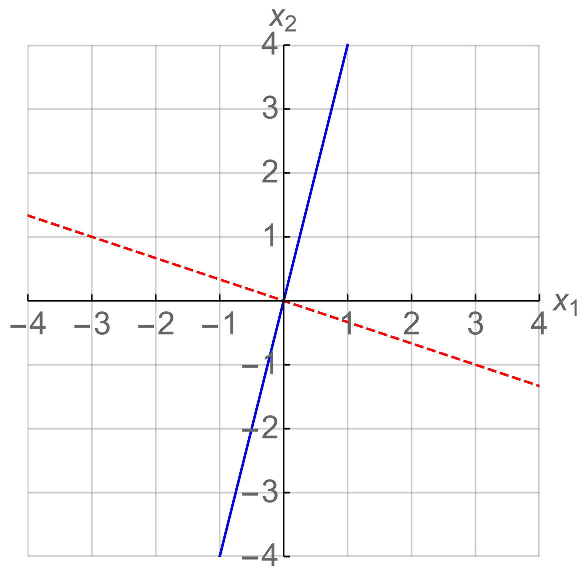
**TRUE** **FALSE**

## Solution.

**a)** True. Recall that when we say a matrix has two pivots, we mean that its RREF has two pivots. $\text{rank}(A)$ is the same as number of pivots in $A$. $\dim(\text{Nul}A)$ is the same as the number of free variables. Moreover by the Rank Theorem, $\text{rank}(A) + \dim(\text{Nul}A) = 10$, so $\dim(\text{Nul}A) = 10 - 2 = 8$.

**b)** True. Because $a$ and $b$ are independent, $a + b$ and $a$ are linearly independent, and furthermore $a$ and $b$ are in $\text{Span}\{a, a + b\}$. Next, $c$ is independent from $\{a, b\}$, so $b + c$ is independent from $\{a, a + b\}$, meaning that $\{a, a + b, b + c\}$ is independent by the increasing span criterion. Since $a, a + b, b + c$ are all clearly in $\text{Span}\{a, b, c\}$, by the basis theorem $\{a, a + b, b + c\}$ also form a span for $\text{Span}\{a, b, c\} = V$. Alternatively, we could notice that $a$, $b$, and $c$ are $\text{Span}\{a, a+b, b+c\}$, and since $V = \text{Span}\{a, b, c\}$ it is a three-dimensional space spanned by the set of three elements $\{a, a+b, b+c\}$, those three elements must form a basis, by the basis theorem.

**3.** Write a matrix $A$ so that Col($A$) is the solid blue line and Nul($A$) is the dotted red line drawn below.




## Solution.

We'd like to design an $A$ with the prescribed column space Span $\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$ and null space Span $\left\{\begin{pmatrix}3\\-1\end{pmatrix}\right\}$.

We start with analyzing the null space. We can write parametric form of the null space:

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = t\begin{pmatrix}3\\-1\end{pmatrix} \quad \text{is the same as} \quad \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-3x_2\\x_2\end{pmatrix}$$

Then this implies the RREF of $A$ must be $\begin{pmatrix}1 & 3\\0 & 0\end{pmatrix}$.

Now we need to combine the information that column space is Span $\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$. That means the second row must be 4 multiple of the first row. Therefore the second row must be $\begin{pmatrix}4 & 12\end{pmatrix}$. We conclude,

$$A = \begin{pmatrix}1 & 3\\4 & 12\end{pmatrix}$$

Note any nonzero scalar multiple of the above matrix is also a solution.

**4.** Let $A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix}$, and let $T$ be the matrix transformation associated to $A$, so $T(x) = Ax$.

**a)** What is the domain of $T$? What is the codomain of $T$? Give an example of a vector in the range of $T$.

**b)** This is extra practice in case the studio finishes the rest of the worksheet early. The RREF of $A$ is $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(i) Write bases for Col($A$) and Nul($A$).

(ii) Is there a vector in the codomain of $T$ which is not in the range of $T$? Justify your answer.

## Solution.

**a)** The domain is $\mathbf{R}^4$; the codomain is $\mathbf{R}^3$. The vector $0 = T(0)$ is contained in the range, as is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

**b)** (i) First, recall that the columns of $A$, which correspond to pivots in the RREF of $A$, form a basis for Col($A$). We note that the first and second column of the RREF of $A$ contain pivots. Therefore, the first and second columns of $A$ form a basis for Col($A$). That is,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Notice that the columns in the RREF of $A$ do not form such basis themselves and in order to write a basis for Col($A$), we need to use the corresponding columns in the matrix $A$ itself.

In order to write a basis for Nul($A$), we need to find the solution to the matrix equation $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$ in parametric vector form. Since $x_3$ and $x_4$ are free variables in this example. The parametric solution would be

$$\begin{cases} x_1 = -3x_3 - x_4 \\ x_2 = -x_3 - x_4 \end{cases},$$

which in vector form can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul($A$) is given by

$$\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(ii) Yes. The range of $T$ is the column span of $A$, and $A$ only has two pivots, so its column span is a 2-dimensional subspace of $\mathbf{R}^3$. Since $\dim(\mathbf{R}^3) = 3$, the range is not equal to $\mathbf{R}^3$.