

## Math 1553 Worksheet §2.3, S2.4

### Solutions

1. True or false. If the statement is *always* true, answer true. Otherwise, answer false and give an example to show it is false.
  - a) Suppose  $A$  is an  $m \times n$  matrix and  $b$  is a vector in  $\mathbf{R}^m$ . If  $Ax = b$  is inconsistent, then  $A$  does not have a pivot in every column.
  - b) If  $A$  is a  $4 \times 3$  matrix, then the equation  $Ax = b$  is inconsistent for some  $b$  in  $\mathbf{R}^4$ .
  - c) Suppose  $A$  is a  $3 \times 3$  matrix with two pivots, and suppose that  $b$  is a vector so that  $Ax = b$  is consistent. Then the solution set for  $Ax = b$  is a plane.

### Solution.

- a) False, for instance consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This is an inconsistent system even though  $A$  has a pivot in each column.

- b) True. Any  $4 \times 3$  matrix  $A$  will have at most 3 pivots, so  $A$  cannot have a pivot in every row. For example, consider the augmented matrix  $(A \mid b)$  below.

$$\left( \begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array} \right)$$

- c) False. The matrix  $A$  has two pivots, which means that the *column span* of  $A$  is plane, but this is not what the question is asking! It is asking about the solution set to  $Ax = b$ , not the column span of  $A$ .

Since  $Ax = b$  corresponds to a system of 3 equations in 3 variables, the fact that  $A$  has two pivots means that the system will have exactly one free variable, so the solution set will be a line in  $\mathbf{R}^3$ .

2. Let  $A = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$ . On the same graph, draw each of the following:

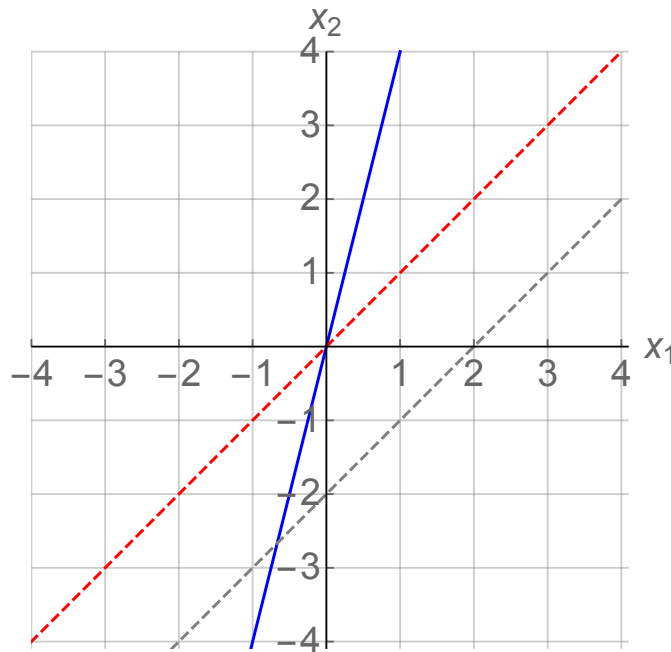
(a) The span of the columns of  $A$ .

(b) The set of solutions to  $Ax = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ .

(c) The set of solutions to  $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ .

Label each of these clearly.

**Solution.**



The blue line is the span of columns of  $A$ :  $\text{Span} \left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$ . If you draw the two column vectors, you will see they both fall on the line  $x_2 = 4x_1$ .

The red dashed line is the span of solutions of  $Ax = 0$ :  $\text{Span} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$ . To see this is the case, you can row reduce the augmented matrix to RREF, which is  $\left( \begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array} \right)$ . That implies the solution set is the line  $x_2 = x_1$ .

The gray dashed line is the set of solutions to  $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$ . To see this is the case, you can row reduce the corresponding augmented matrix to RREF, which is  $\left( \begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array} \right)$ . That implies the solution set is the line  $x_1 = 2 + x_2$  (where  $x_2$  is free) which yields

parametric form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

In other words, this solution set is the line through  $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$  parallel to the span of  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ .

3. Find the set of solutions to  $x_1 - 3x_2 + 5x_3 = 0$  and write your answer in parametric vector form. Next, find the set of solutions to  $x_1 - 3x_2 + 5x_3 = 3$  and write the solutions in parametric vector form. How do the solution sets compare geometrically?

**Solution.**

The homogeneous system  $x_1 - 3x_2 + 5x_3 = 0$  corresponds to the augmented matrix  $(1 \ -3 \ 5 \ | \ 0)$ , which has two free variables  $x_2$  and  $x_3$ .

$$x_1 = 3x_2 - 5x_3 \quad x_2 = x_2 \text{ (free)} \quad x_3 = x_3 \text{ (free)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

The solution set for  $x_1 - 3x_2 + 5x_3 = 0$  is the plane spanned by  $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$ .

The nonhomogeneous system  $x_1 - 3x_2 + 5x_3 = 3$  corresponds to the augmented matrix  $(1 \ -3 \ 5 \ | \ 3)$  which has two free variables  $x_2$  and  $x_3$ .

$$x_1 = 3 + 3x_2 - 5x_3 \quad x_2 = x_2 \quad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This solution set (red) is the *translation* by  $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$  of the plane (green) spanned by  $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$  and  $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$ .

