Math 1553 Worksheet §1.2, §1.3

Solutions

1. **a)** Circle the 'operations' that are legal to use in row reduction, in other words, the operations that will not change the solution set of an arbitrary linear system.

(1) $R_2 = R_3 + 4R_2$

(2) $R_3 = 3R_3$

(3) $R_1 = R_2 - R_3$

(4) $R_1 \longleftrightarrow R_2$

(5) $R_2 = R_2 + (R_1)^5$

(6) $R_3 = R_3 - \ln(R_2)$

b) These are row operations only. Try performing a column operation: for example, try doubling any column in $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$. What happens to the solution set?

Solution.

(a) Only (1), (2), (4) are legit operations in row reduction.
(3) $R_1 = R_2 - R_3$ is not because it removed the R_1, and it will lose the information in R_1.
(5), (6) have nonlinear operations $(R_1)^5, \ln R_2$.

(b) The solution set for $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$ is $x = 1$. Doubling any column in $\left(\begin{array}{c|c} 1 & 1 \end{array}\right)$ either changes the augmented matrix to $\left(\begin{array}{c|c} 2 & 1 \end{array}\right)$ or $\left(\begin{array}{c|c} 1 & 2 \end{array}\right)$ corresponding to (different) solution sets $x = \frac{1}{2}$ and $x = 2$. You cannot perform column operations, as that will change the solution set of the linear system.

2. **a)** Which of the following matrices are in row echelon form (REF)? Which are in reduced row echelon form (RREF)?

b) For the matrices that are in REF or RREF, which entries are the pivots? What are the pivot columns?

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{cccc|c} 1 & 1 & 3 & 1 & 1 \\ 0 & 0 & 4 & 2 & 2 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 2 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

c) Why is RREF useful, i.e. what information does it reveal about the linear system?

d) How many nonzero entries are there in a pivot column of a matrix that is in RREF?

Solution.

a) The first is in reduced row echelon form; the second is in row echelon form. The third is neither.

b) The pivots are in red; the other entries in the pivot columns are in blue. The third is not in REF, but with one swap $R_2 \longleftrightarrow R_3$ it will be REF and pivots are easy to find.

c) One reason why RREF is useful is that it tells us whether a system is consistent. Namely, if the augmented matrix's RREF has a pivot in the rightmost column, then the system is inconsistent; if not, then it is consistent.

d) In a pivot column of RREF, we will have to clear all entries above and below the pivot. This means it has only 1 nonzero entry.

3. Each matrix below is in RREF. In each case, determine whether the corresponding system of linear equations is consistent, and if so, how many solutions does it have?

$$\text{(a)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{array}\right), \quad \text{(b)} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 5 \end{array}\right), \quad \text{(c)} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 7 \end{array}\right), \quad \text{(d)} \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Solution.

a) [1 solution]. $x_1 = 1, \quad x_2 = 2, \quad x_3 = 3.$

b) [no solution]. There is a pivot in the rightmost column.

c) [infinitely many solutions]. x_2 is a free variable.

d) [infinitely many solutions]. Every point (x, y, z) in $\mathbb{R}^3$ satisfies $0x+0y+0z = 0$.

4. Find the parametric form for the solution set of the following system of linear equations in x_1, x_2, and x_3 by putting an augmented matrix into reduced row echelon form. State which variables (if any) are free variables. Describe the solution set geometrically.

$$\begin{aligned} x_1 + 3x_2 + x_3 &= 1 \\ -4x_1 - 9x_2 + 2x_3 &= -1 \\ - 3x_2 - 6x_3 &= -3. \end{aligned}$$

Solution.

$$\left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ -4 & -9 & 2 & -1 \\ 0 & -3 & -6 & -3 \end{array}\right) \xrightarrow{R_2=R_2+4R_1} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & -3 & -6 & -3 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+R_2} \left(\begin{array}{ccc|c} 1 & 3 & 1 & 1 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-R_2} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 3 & 6 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{R_2=R_2\div 3} \left(\begin{array}{ccc|c} 1 & 0 & -5 & -2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

The variables x_1 and x_2 correspond to pivot columns, but x_3 is free.

$$x_1 = -2 + 5x_3, \qquad x_2 = 1 - 2x_3, \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

This consistent system in three variables has one free variable, so the solution set is a line in $\mathbf{R}^3$.