Math 1553 Worksheet §5.4-5.6

1. True or false. Justify your answer.

a) A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

b) It is possible for a 2×2 stochastic matrix to have $-i/2$ as an eigenvalue.

Solution.

a) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible since A is 3×3.

b) No. The matrix must have $\lambda = 1$ as an eigenvalue since it is stochastic, but if $\lambda = -i/2$ is an eigenvalue then so is $\lambda = i/2$, which is impossible since a 2×2 matrix cannot have more than two eigenvalues.

2. Let $A = \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 1 \end{pmatrix}^{-1}$, and let $x = \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. What happens to $A^n x$ as n gets very large?

Solution.

We are given diagonalization of A, which gives us the eigenvalues and eigenvectors.

$$\begin{aligned} A^n x &= A^n \left(\begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right) = A^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + A^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= 1^n \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \left(\frac{1}{2} \right)^n \begin{pmatrix} 3 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -1 \end{pmatrix} + \begin{pmatrix} \frac{3}{2^n} \\ \frac{1}{2^n} \end{pmatrix}. \end{aligned}$$

As n gets very large, the entries in the second vector above approach zero, so $A^n x$ approaches $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. For example, for $n = 15$,

$$A^{15} x \approx \begin{pmatrix} 2.00009 \\ -0.999969 \end{pmatrix}.$$

3. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector.

Solution.

The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue $\lambda = 1-2i$, we use the shortcut trick you may have seen in class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A - \lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\left(A-(1-2i)I \mid 0\right) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array}\right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1+2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

If you used row-reduction for finding eigenvectors, you would find $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1-2i$, and $w = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1+2i$.

4. Axel and Billy are magicians who compete for customers in a group of 180 people. Today, Axel has 120 customers and Billy has 60 customers. Each day:

- 30% of Axel's customers keep attending Axel's show, while 70% of Axel's customers switch to Billy's show.
- 80% of Billy's customers attend Billy's show, while 20% of Billy's customers switch to Axel's show.

(a) Write a positive stochastic matrix B and a vector x so that Bx will give the number of customers for Axel's show and Billy's show (in that order) tomorrow. You do not need to compute Bx.

Solution.

Using the information we have been given, we have

$$B = \begin{pmatrix} 0.3 & 0.2 \\ 0.7 & 0.8 \end{pmatrix}, \qquad x = \begin{pmatrix} 120 \\ 60 \end{pmatrix}.$$

(b) Find the steady-state vector w for B.

Solution.

$\left(B - I \mid 0 \right) = \left(\begin{array}{cc|c} -0.7 & 0.2 & 0 \\ 0.7 & -0.2 & 0 \end{array} \right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -2/7 & 0 \\ 0 & 0 & 0 \end{array} \right)$, so for the 1-eigenspace we get $x_1 = \frac{2}{7}x_2$ where x_2 is free, thus the vector $v = \begin{pmatrix} 2/7 \\ 1 \end{pmatrix}$ spans the 1-eigenspace. The steady state vector is

$$w = \frac{1}{2/7+1}\begin{pmatrix} 2/7 \\ 1 \end{pmatrix} = \frac{1}{9/7}\begin{pmatrix} 2/7 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/9 \\ 7/9 \end{pmatrix}.$$

(c) In the long run, roughly how many daily customers will Billy have?

Solution.

From the steady-state vector, Billy will have roughly 7/9 of the 180 total customers, which is $\dfrac{7}{9} \cdot 180 = 140$ customers.