Math 1553 Worksheet §6.1 - §6.5

Solutions

1. True/False. Justify your answer.

a) If u is a vector that is orthogonal to itself, then $u = 0$.

b) If y is in a subspace W, the orthogonal projection of y onto $W^\perp$ is 0.

c) If x is orthogonal to v and w, then x is also orthogonal to $v - w$.

Solution.

(1) TRUE: If u is orthogonal to itself, then $u \cdot u = ||u||^2 = 0$. Therefore, u has length 0, so $u = 0$.

(2) TRUE: y is in W, so $y \perp W^\perp$. Its orthogonal projection onto W is y and orthogonal projection onto $W^\perp$ is 0. In fact y has orthogonal decomposition $y = y + 0$, where y is in W and 0 is in $W^\perp$.

(3) TRUE: By properties of the dot product, if x is orthogonal to v and w then x is orthogonal to everything in Span$\{v, w\}$ (which includes $v - w$).

2. **a)** Find the standard matrix B for proj_W, where $W = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}\right\}$.

b) What are the eigenvalues of B? Is B diagonalizable?

c) Let $x = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$. Find the projection x_W of x onto the subspace W and the orthogonal projection $x_{W^\perp}$ of x onto the subspace $W^\perp$.

Solution.

a) We use the formula $B = \dfrac{1}{u \cdot u} uu^T$ where $u = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ (this is the formula $B = A(A^TA)^{-1}A^T$ when "A" is just the single vector u).

$$B = \frac{1}{1(1) + 1(1) + (-1)(-1)} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & -1 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

$$\implies B = \frac{1}{3} \begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

b) $Bx = x$ for every x in W, and $Bx = 0$ for every x in $W^\perp$, so B has two eigenvalues: $\lambda_1 = 1$ with algebraic and geometric multiplicity 1, $\lambda_2 = 0$ with algebraic and geometric multiplicity 2. Therefore, B is diagonalizable. As an aside, we could actually compute B using diagonalization if we wanted! Here

$v_1 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is an eigenvector for $\lambda_1 = 1$, whereas $v_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ are linearly independent vectors that are orthogonal to v_1, so they span the eigenspace for $\lambda_2 = 0$. Therefore

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ -1 & 0 & 1 \end{pmatrix}^{-1}$$

c) Now that we've computed our standard matrix B for proj_W, we can represent the projection x_W of x onto $W = \mathrm{Span}\left\{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right\}$ as

$$x_W = Bx = \frac{1}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

and thus the orthogonal projection $x_{W^\perp}$ is just whatever is left of x after we subtract x_W (the part of x that lies on W):

$$x_{W^\perp} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

3. Use least-squares to find the best fit line $y = Ax + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$.

Solution.

We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= A(0) + B \\ 8 &= A(1) + B \\ 8 &= A(3) + B \\ 20 &= A(4) + B \end{aligned} \qquad \iff \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

We compute

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right).$$

Hence the least squares solution is $A = 4$ and $B = 1$, so the best fit line is $y = 4x + 1$.