Math 1553 Final Examination, SOLUTIONS, Spring 2025

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A+HP, 8:25-9:15 AM) Jankowski (C, 9:30-10:20 AM)

Al Ahmadieh (I, 2:00-2:50 PM) Al Ahmadieh (M, 3:30-4:20 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- Simplify all fractions as much as possible. As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until 8:50 PM on Tuesday, April 29.

____

1. (2 pts each) **Solutions are on the next page.**

(a) Suppose W is a 2-dimensional subspace of $\mathbf{R}^4$.

If $\begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \end{pmatrix}$ are in W, then $\left\{ \begin{pmatrix} 1 \\ -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 5 \end{pmatrix} \right\}$ must be a basis for W.

● True

○ False

(b) If $T : \mathbf{R}^a \to \mathbf{R}^b$ is a linear transformation that is onto, then $a \geq b$.

● True

○ False

(c) If v is an eigenvector of an $n \times n$ matrix A, then $5v$ must also be an eigenvector of A.

● True

○ False

(d) If A is an invertible $n \times n$ matrix, then $\det(A) = 0$.

○ True

● False

(e) Suppose W is a subspace of $\mathbf{R}^n$ and x is a vector in $\mathbf{R}^n$. If x_W is the orthogonal projection of x onto W, then $x \cdot x_W = 0$.

○ True

● False

Solution: Problem 1

(a) True by the Basis Theorem: since they are two linearly independent vectors in the 2-dimensional subspace W, the two vectors must form a basis for W.

(b) True. The matrix A for T is $b \times a$, and T is onto if and only if A has a pivot in every row. This means that A must be a square matrix or a "wide" matrix, so the number of columns must be at least as large as the number of rows, therefore $a \geq b$.

(c) True. We can check both conditions for eigenvector:

- $5v \neq 0$ since $v \neq 0$.
- $Av = \lambda v$ for some λ, therefore $A(5v) = 5Av = 5\lambda v = \lambda(5v)$.

Therefore, $5v$ is an eigenvector in the same eigenspace as v. Alternatively, we could also have just used the fact that eigenspaces are subspaces to argue $5v$ (which we know is not the zero vector) is an eigenvector.

(d) False: a standard fact about determinants is that A is invertible if and only if $\det(A) \neq 0$.

(e) False. This was copied from the Studypalooza problems list. For example, if W is the span of e_1 and $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, then $x_W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and

$$x \cdot x_W = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot 10 = 1.$$

In fact, if x is in W, then $x = x_W$, so $x \cdot x_W = x \cdot x = ||x||^2$ which is never zero unless x is the zero vector.

2. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (3 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; xy = 0 \right\}$. Determine which properties of a subspace are satisfied by V. Clearly fill in the bubble for all that apply.

● V contains the zero vector of $\mathbf{R}^2$.

○ V is closed under addition. In other words, if u and v are vectors in V, then $u + v$ must be in V.

● V closed under scalar multiplication? In other words, if u is a vector in V and c is a real number, then cu must be in V.

(b) (2 points) Which **one** of these is a 2-dimensional subspace of $\mathbf{R}^4$?

○ $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y + z - w = 1 \right\}$

○ Row $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

○ Nul $\begin{pmatrix} 1 & -3 & 4 & 9 \end{pmatrix}$

● $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ -2 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 2 \\ 3 \end{pmatrix} \right\}$

(c) (3 points) Suppose A is a 60×80 matrix whose RREF has exactly 45 pivots. Which of the following statements are true? Fill in the bubble for all that apply.

○ The null space of A is a 35-dimensional subspace of $\mathbf{R}^{60}$.

● The row space of A is a 45-dimensional subspace of $\mathbf{R}^{80}$.

● Let $W = \text{Col}(A)$, and let B be the matrix for orthogonal projection onto W. Then the rank of B is 45.

(d) (2 points) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be the linear transformation that satisfies

$$T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -3 \\ 2 \\ 3 \end{pmatrix}, \quad T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \text{ and } \quad T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \quad \text{Find } T\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Fill in the bubble for your answer below.

○ $\begin{pmatrix} 0 \\ 2 \\ 4 \end{pmatrix}$ ○ $\begin{pmatrix} -1 \\ 6 \\ 10 \end{pmatrix}$ ● $\begin{pmatrix} -3 \\ 6 \\ 10 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ ○ not enough info

Solution: Problem 2

(a) This was essentially copied from the 2.6 Webwork but made easier.

- Yes, V contains $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since $0 \cdot 0 = 0$.
- No, V is not closed under addition. For example, $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are in V, but $u + v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ which is not in V since $1 \cdot 1 \neq 0$.
- Yes, V is closed under scalar multiplication. If $\begin{pmatrix} x \\ y \end{pmatrix}$ is in V then so is $\begin{pmatrix} cx \\ cy \end{pmatrix}$ since $(cx)(cy) = c^2xy = c^2(0) = 0$.

Alternatively, we could have drawn V to do the problem. It is the union of the x-axis and the y-axis in $\mathbf{R}^2$.

(b) Option (i) is not a subspace since the equation is set to 1, so it does not include the zero vector.

Option (ii) is not even a subspace of $\mathbf{R}^4$ since the row space of any 4×3 matrix is a subspace of $\mathbf{R}^3$, not $\mathbf{R}^4$.

Option (iii) is a subspace of $\mathbf{R}^4$, but it is 3-dimensional because A has one pivot and thus there are 3 variables in the solution set to $Ax = 0$.

Option (iv) is a 2-dimensional subspace of $\mathbf{R}^4$, since it lives in $\mathbf{R}^4$ and its first and third columns are its only pivot columns.

(c) Option (i) is false: $\text{Nul}(A)$ is a subspace of $\mathbf{R}^{80}$, not $\mathbf{R}^{60}$.

Option (ii) is true: since A has 45 pivots and $\dim(\text{Row } A) = \dim(\text{Col } A)$, the row space of A is 45-dimensional, and it lives in $\mathbf{R}^{80}$ since A has 80 columns.

Option (iii) is true: $W = \text{Col}(A)$ is 45-dimensional, and the column space of B is just W, so $\dim(\text{Col } B) = 45$.

(d) $T\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = 3T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 2T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + T\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -9 \\ 6 \\ 9 \end{pmatrix} + \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ 10 \end{pmatrix}$.

3. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (3 points) Let A be an $n \times n$ matrix. Which of the following statements are true? Fill in the bubble for all that apply.

● If there is an $n \times n$ matrix B so that $AB = I$, then A must be invertible.

○ If $\lambda = 1$ is an eigenvalue of A, then $\text{Nul}(A - I) = \{0\}$.

● If the equation $Ax = b$ is inconsistent for some b in $\mathbf{R}^n$, then the homogeneous equation $Ax = 0$ must have infinitely many solutions.

(b) (3 points) Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with standard matrix A, so $T(x) = Ax$. Which of the following statements guarantee that T is onto? Fill in the bubble for all that apply.

● For every y in $\mathbf{R}^m$, there is at least one x in $\mathbf{R}^n$ so that $T(x) = y$.

○ For every x in $\mathbf{R}^n$, there is at least one y in $\mathbf{R}^m$ so that $T(x) = y$.

● The equation $Ax = b$ is consistent for each b in $\mathbf{R}^m$.

(c) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first rotates vectors by 90 degrees clockwise, then reflects vectors across the line $y = x$.

Find $T\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. Fill in the bubble for your answer below.

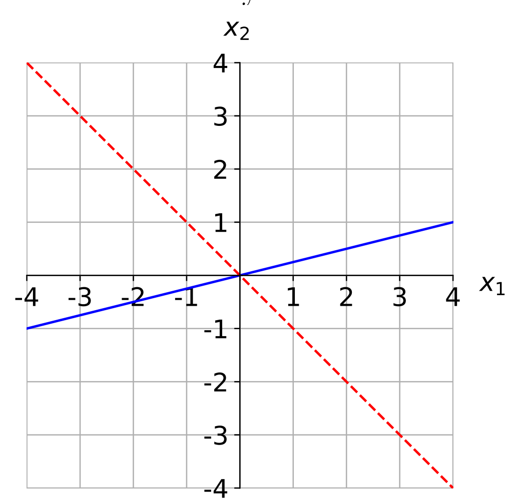
○ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ● $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

(d) (2 points) Select the matrix whose column space is the solid line and whose null space is the dashed line drawn below. Fill in the bubble for your answer.

○ $\begin{pmatrix} 1 & 1 \\ -4 & 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 1 \\ 2 & -4 \end{pmatrix}$

● $\begin{pmatrix} 1 & -1/2 \\ -4 & 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 1/2 \\ -4 & -2 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 2 \\ -4 & -8 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & -2 \\ -4 & 8 \end{pmatrix}$



Solution: Problem 3

(a) Option (i) is true directly from the Invertible Matrix Theorem.

Option (ii) is not true: if $\lambda = 1$ is an eigenvalue of A, then $\mathrm{Nul}(A - I)$ contains infinitely many vectors.

Option (iii) is true: if $Ax = b$ is inconsistent for some b in $\mathbf{R}^n$, then A must have fewer than n pivots. Therefore, at least one column in A does not have a pivot, so $Ax = 0$ has infinitely many solutions.

(b) Option (i) is nearly verbatim the definition of onto.

Option (ii) is a modification of the definition of transformation and does not guarantee that T is onto. For example, it is satisfied by the transformation $T(a, b) = (a, 0)$ which is not onto.

Option (iii) is also basically the definition of onto.

(c) Rotating $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ by 90° clockwise gives us $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, then reflecting across $y = x$ gives us $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

Alternatively, we could do this problem with matrix multiplication. Remember that the first transformation's matrix goes on the right (because we multiply this by the vector first) and the second transformation's matrix is on the left:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

(d) Call the matrix A. We are told that $\mathrm{Col}(A)$ is the span of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$, so each column must be a scalar multiple of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$. Also, $\mathrm{Nul}(A)$ is the span of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, so $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. There is only one matrix that satisfies both conditions, namely $\begin{pmatrix} 1 & -1/2 \\ -4 & 2 \end{pmatrix}$ since

$$\begin{pmatrix} 1 & -1/2 \\ -4 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 - 1 \\ -4 + 4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

4. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Find the area of the triangle with vertices $(0,1)$, $(1,3)$, and $(2,-4)$.

○ 1/2 ○ 1 ○ 3/2 ○ 5/2 ○ 7/2

● 9/2 ○ 3 ○ 5 ○ 9 ○ none of these

(b) (2 points) Suppose A and B are 2×2 matrices satisfying $\det(A) = 3$ and $\det(2AB^{-1}) = 1$. Find $\det(B)$.

○ 3 ○ 6 ○ 9 ● 12 ○ 18 ○ 1/6

○ 1/9 ○ 1/12 ○ 1/18 ○ not enough info to find $\det(B)$

(c) (2 points) Let A be an 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(1-\lambda)(5-\lambda).$$

Which **one** of the following statements must be true?

● There is a nonzero vector v in $\mathbf{R}^3$ with the property that v is **not** an eigenvector of A.

○ A is invertible.

○ Every nonzero vector in $\mathbf{R}^3$ is an eigenvector of A.

○ If u and v are eigenvectors of A, then $u+v$ must also be an eigenvector of A.

(d) (4 points) Suppose A and B are $n \times n$ matrices. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

● If 0 is an eigenvalue of B, then 0 must also be an eigenvalue of BA.

○ If A and B are invertible, then $(AB)^{-1} = A^{-1}B^{-1}$.

● If v is a vector in the null space of B, then v must also be in the null space of AB.

○ If w is a vector in the column space of A, then w must also be in the column space of AB.

Solution: Problem 4

(a) The vector from $(0,1)$ to $(1,3)$ is $v_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

The vector from $(0,1)$ to $(2,-4)$ is $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$. The area of the triangle is

$$\frac{1}{2}\left|\det\begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}\right| = \frac{1}{2}|-5-4| = 9/2.$$

(b) Multiplying a matrix by 2 means multiplying each row by 2, so for a 2×2 matrix this multiplies the determinant by $2^2 = 4$. Using the properties of determinants, we have

$$1 = \det(2AB^{-1}) = 2^2 \det(A)\det(B^{-1}) = 4 \cdot 3 \cdot \frac{1}{\det(B)}.$$

Therefore $\det(B) = 12$.

(c) Option (ii) is false: $\lambda = 0$ is an eigenvalue of A, so A is NOT invertible.

Options (iii) and (iv) are false because if u and v are eigenvectors in **different** eigenspaces of A, then their sum is **never** an eigenvector of A. For example, if u and v are nonzero vectors with $Au = u$ and $Av = 5v$, then $A(u+v) = u + 5v$ which is not a scalar multiple of $u + v$.

Option (i) is correct for the same reason just given above. Take a nonzero u in the 1-eigenspace of A and a nonzero v in the 5-eigenspace of A. Then u and v are automatically linearly independent since they are in different eigenspaces, so $u + v$ is nonzero. Also, $u + v$ is not an eigenvector of A since

$$A(u+v) = u + 5v \text{ which is not a scalar multiple of } u \text{ or } v..$$

(d) Option (i) is true: if 0 is an eigenvalue of B, then $\det(BA) = \det(B)\det(A) = 0 \cdot \det(A) = 0$. Therefore, BA is not invertible, which means that $\lambda = 0$ is an eigenvalue of BA.

Option (ii) is not true: in reality, $(AB)^{-1} = B^{-1}A^{-1}$.

Option (iii) is true: If $Bv = 0$ then $ABv = A(0) = 0$, so v is in the null space of AB.

Option (iv) is not true: for example, take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is in $\mathrm{Col}(A)$, but $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not in $\mathrm{Col}(AB)$.

5. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Which **one** of the following matrices has no real eigenvalues? Clearly fill in the bubble for your answer.

○ $A = \begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 1 & 5 \end{pmatrix}$

○ The 2×2 matrix A that reflects vectors across the line $y = -x$, then rotates vectors by 90° counterclockwise.

● The 2×2 matrix A that rotates vectors by 60° clockwise.

(b) (2 points) Let $A = \begin{pmatrix} 0.9 & 0.5 \\ 0.1 & 0.5 \end{pmatrix}$. Find the steady-state vector for A.
Fill in the bubble for your answer below.

○ $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ ○ $\begin{pmatrix} 5/9 \\ 4/9 \end{pmatrix}$ ○ $\begin{pmatrix} 9/14 \\ 5/14 \end{pmatrix}$ ○ $\begin{pmatrix} 5/14 \\ 9/14 \end{pmatrix}$ ○ $\begin{pmatrix} 9/10 \\ 1/10 \end{pmatrix}$

● $\begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}$ ○ $\begin{pmatrix} 1/6 \\ 5/6 \end{pmatrix}$ ○ $\begin{pmatrix} 1/10 \\ 9/10 \end{pmatrix}$ ○ none of these

(c) (3 points) Let A be the 2×2 matrix that reflects each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = -10x$. Which of the following statements are true? Fill in the bubble for all that apply.

○ $A\begin{pmatrix} 1 \\ 10 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

○ The eigenvalues of A are 0 and 1.

● The 1-eigenspace of A is the span of $\begin{pmatrix} 1 \\ -10 \end{pmatrix}$.

(d) (3 points) Let W be a 4-dimensional subspace of $\mathbf{R}^6$, and let B be the matrix for orthogonal projection onto W. Which of the following statements must be true? Fill in the bubble for all that apply.

● The eigenvalues of B are 0 and 1.

● If a vector x is in W and $W^\perp$, then x must be the zero vector.

● If x is in $\mathbf{R}^6$, then Bx is the closest vector to x in W.

Solution: Problem 5

(a) Option (v) has no real eigenvalues, because rotating any nonzero vector by 60° clockwise will **never** give you a scalar multiple of the original vector (it is on a different line through the origin).

Option (i) has eigenvalues 1 and 5; option (ii) is 3×3 so it must have at least one real eigenvalue because every "odd $\times$ odd" matrix has at least one real eigenvalue due to the fact that complex eigenvalues come in conjugate pairs; (iii) has eigenvalues 0 and 2; and (iv) has eigenvalues 1 and -1 since it corresponds to the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ (alternatively, we could just note that $T(e_1) = e_1$ so 1 is an eigenvalue).

(b) We find $\left(A - I \mid 0 \right) = \left(\begin{array}{cc|c} -0.1 & 0.5 & 0 \\ 0.1 & -0.5 & 0 \end{array} \right) \to \left(\begin{array}{cc|c} 1 & -5 & 0 \\ 0 & 0 & 0 \end{array} \right)$. This gives us $x_1 = 5x_2$, so 1-eigenspace is the span of $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$. The steady-state w is

$$w = \frac{1}{5+1} \begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 5/6 \\ 1/6 \end{pmatrix}.$$

(c) Option (i) is false: 0 is not an eigenvalue of A, so $Ax = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ only has the trivial solution, thus $A \begin{pmatrix} 1 \\ 10 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Option (ii) is false: we have seen many times that the eigenvalues for a 2×2 reflection matrix are -1 and 1, not 0 and 1.

Option (iii) is true: the 1-eigenspace of A is the line $y = -10x$, which is the span of $\begin{pmatrix} 1 \\ -10 \end{pmatrix}$.

(d) Option (i) is true: it is a standard fact for orthogonal projections (aside of the zero matrix and identity matrix) that the eigenvalues of B are 0 and 1.

Option (ii) is true: in such a case, x would be orthogonal to itself, so

$$0 = x \cdot x = ||x||^2$$

which means $x = 0$. I think we took this one directly from the Chapter 6 webwork.

Option (iii) is true. It is a fundamental fact about orthogonal projections.

6. Multiple choice. You do not need to show your work on this page, and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 2 & 6 \\ -1 & 1 \end{pmatrix}$.

○ 1 and 4 ○ $3 \pm 4i$ ○ $-3 \pm 4i$ ○ $\frac{5}{2} \pm i\frac{\sqrt{15}}{2}$ ○ $2 \pm 3i$

○ $\frac{-5}{2} \pm i\frac{\sqrt{15}}{2}$ ○ $\frac{-3}{2} \pm i\frac{\sqrt{23}}{2}$ ● $\frac{3}{2} \pm i\frac{\sqrt{23}}{2}$ ○ $\frac{1}{2} \pm i\frac{\sqrt{21}}{2}$

○ none of these

(b) (2 points) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

Determine which **one** of the following vectors is in $W^{\perp}$. Fill in the bubble for your answer.

○ $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ ● $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ -1 \\ 1 \end{pmatrix}$

(c) (4 points) Let A be an $m \times n$ matrix. Which of the following statements must be true? Fill in the bubble for all that apply.

● $(\text{Col } A)^{\perp} = \text{Nul}(A^T)$

○ If $m > n$, then the matrix transformation $T(x) = Ax$ cannot be one-to-one.

● $\text{Row } A = (\text{Nul } A)^{\perp}$

● If the homogeneous equation $Ax = 0$ has only the trivial solution, then $\dim(\text{Col } A) = n$.

(d) (2 points) Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 10y - 2z = 0 \right\}$.

Which **one** of the following describes $W^{\perp}$? Fill in the bubble for your answer.

○ $\text{Span}\left\{ \begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix} \right\}$ ● $\text{Span}\left\{ \begin{pmatrix} 1 \\ -10 \\ -2 \end{pmatrix} \right\}$ ○ $\text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

○ $\text{Col}\begin{pmatrix} 1 & -10 & -2 \end{pmatrix}$ ○ $\text{Nul}\begin{pmatrix} 1 \\ -10 \\ -2 \end{pmatrix}$

Solution: Problem 6

(a) The formula $\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$ gives us

$$\lambda^2 - 3\lambda + 8 = 0, \quad \text{so} \quad \lambda = \frac{3 \pm \sqrt{(-3)^2 - 4(8)}}{2} = \frac{3 \pm i\sqrt{23}}{2}.$$

(b) $W^\perp = \text{Nul}\begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \end{pmatrix}$, and the system $\left(\begin{array}{ccc|c} 1 & 0 & -3 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$ gives us $x_1 = 3x_3$ and $x_2 = -x_3$, where x_3 is free. Therefore,

$$W^\perp = \text{Span}\left\{ \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \right\}.$$

We could have instead just checked everything by hand, by taking the other answer choices and computing their dot product with each of the basis vectors for W. We'd get at least one nonzero result each time for the other answer choices. However,

$$\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = 3 - 3 = 0, \qquad \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 0 - 1 + 1 = 0.$$

(c) Option (i) is true: it is a standard fact from section 6.2.

Option (ii) is false because it is possible for T to be one-to-one if $m > n$. For example, T is one-to-one if A is the 3×2 matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Option (iii) is true: it is a standard fact from section 6.2.

Option (iv) is true by the Rank Theorem: if $Ax = 0$ has only the trivial solution then

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = n, \qquad \dim(\text{Col } A) + 0 = n, \qquad \dim(\text{Col } A) = n.$$

(d) We basically copied and pasted a problem from the 6.2 Webwork. You can do this without any work. Here $W = \text{Nul}(A)$ for the matrix $A = \begin{pmatrix} 1 & -10 & -2 \end{pmatrix}$, so

$$W^\perp = \text{Nul}(A)^\perp = \text{Row}(A) = \text{Span}\left\{ \begin{pmatrix} 1 \\ -10 \\ -2 \end{pmatrix} \right\}.$$

In fact, since W is a 2-dimensional subspace of $\mathbf{R}^3$, we know $W^\perp$ is a 1-dimensional subspace of $\mathbf{R}^3$. But there is only one answer choice with this property! Alternatively, we could have computed $W^\perp$ and would have found it is the span of $\begin{pmatrix} -1/2 \\ 5 \\ 1 \end{pmatrix}$ which is the same as our answer.

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may result in little or no credit. Parts (a) and (b) are unrelated.

(a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors in $\mathbf{R}^2$ across the line $y = x$, and let $U : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by

$$U(x, y) = (2x + 5y,\ 3x - y,\ x + 7y).$$

i. (2 points) Find the standard matrix A for T. Enter your answer below.

$$A = \begin{pmatrix} T\begin{pmatrix}1\\0\end{pmatrix} & T\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

ii. (3 points) Find the standard matrix B for U. Enter your answer below.

$$B = \begin{pmatrix} U\begin{pmatrix}1\\0\end{pmatrix} & U\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} = \begin{pmatrix}2 & 5\\ 3 & -1\\ 1 & 7\end{pmatrix}$$

iii. (3 points) Find the standard matrix C for $U \circ T$. Enter your answer below.

The matrix is $C = BA = \begin{pmatrix}2 & 5\\ 3 & -1\\ 1 & 7\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}5 & 2\\ -1 & 3\\ 7 & 1\end{pmatrix}$.

(b) (2 points) This part is unrelated to (a).
Is there some $m \times n$ matrix M with the property that

$$\dim((\text{Row } M)^{\perp}) = 2 \quad \text{and} \quad \dim((\text{Col } M)^{\perp}) = 1?$$

If your answer is yes, write such a matrix M in the space below. If your answer is no, fill in the bubble for "no such M exists." You do not need to show your work or justify your answer on this part.

Yes, there is such an M. For example, if we take a 2×3 matrix with 1 pivot, then:

- $\text{Row}(M)$ will be a 1-dimensional subspace of $\mathbf{R}^3$, so $(\text{Row } M)^{\perp}$ will be 2-dimensional.
- $\text{Col}(M)$ will be a 1-dimensional subspace of $\mathbf{R}^2$, so $(\text{Col } M)^{\perp}$ will be 1-dimensional.

For example, $M = \begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$, so $\text{Row}(M) = \text{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}$ and $\dim((\text{Row } M)^{\perp}) = 2$.

Also, $\text{Col}(M) = \text{Span}\left\{\begin{pmatrix}1\\0\end{pmatrix}\right\}$, so $\dim((\text{Col } M)^{\perp}) = 1$.

8. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

   Consider the matrix $A = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix}$.

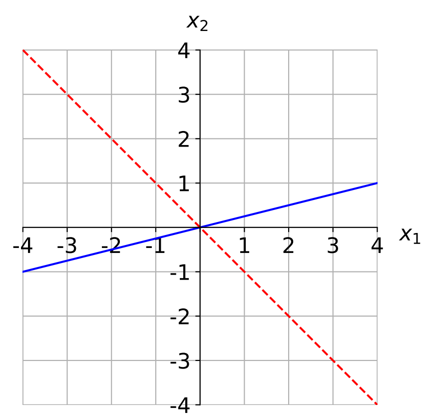
   (a) Find the eigenvalues of A. Write them here: ________________

   $\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$, so

   $$\lambda^2 - (-1)\lambda + (-2-4) = 0, \qquad \lambda^2 + \lambda - 6 = 0, \qquad (\lambda - 2)(\lambda + 3) = 0.$$

   The eigenvalues are $\lambda = 2$ and $\lambda = -3$.

   (b) For each eigenvalue of A, find a basis for the corresponding eigenspace. Draw and clearly label each eigenspace on the graph below.



   Solution: We row-reduce.

   $$\lambda = 2 : \left(A - 2I \mid 0 \right) = \left(\begin{array}{cc|c} -1 & 4 & 0 \\ 1 & -4 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

   This gives $x_1 = 4x_2$ with x_2 free, so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 4x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

   The 2-eigenspace has basis $\left\{ \begin{pmatrix} 4 \\ 1 \end{pmatrix} \right\}$.

   $$\lambda = -3 : \left(A + 3I \mid 0 \right) = \left(\begin{array}{cc|c} 4 & 4 & 0 \\ 4 & 4 & 0 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

   This gives $x_1 = -x_2$ with x_2 free, so $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

   The (-3)-eigenspace has basis $\left\{ \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$.

   We draw the 2-eigenspace as a blue solid line and the (-3)-eigenspace as a red dashed line in the graph above.

(c) A is diagonalizable. In the space provided below, write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.

We need the columns of C to be eigenvectors of A, and for the diagonal entries of D to be the corresponding eigenvalues (in that same order). For example,

$$C = \begin{pmatrix} 4 & -1 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}$$

or

$$C = \begin{pmatrix} -1 & 4 \\ 1 & 1 \end{pmatrix} \text{ and } D = \begin{pmatrix} -3 & 0 \\ 0 & 2 \end{pmatrix}$$

9. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may result in little or no credit.

For this page of the exam, let $W = \text{Span}\left\{\begin{pmatrix}3\\1\end{pmatrix}\right\}$.

(a) i. (2 points) Write one vector v so that the orthogonal projection of v onto W is the zero vector. No work required and no partial credit on this part.

Solution: We are asked to write a v so that $v_W = \begin{pmatrix}0\\0\end{pmatrix}$, which means that v is orthogonal to W. Any vector in $W^\perp$ is correct, for example

$$v = \begin{pmatrix}0\\0\end{pmatrix}, \qquad v = \begin{pmatrix}-1\\3\end{pmatrix}, \qquad v = \begin{pmatrix}1\\-3\end{pmatrix}, \qquad \begin{pmatrix}-2\\6\end{pmatrix}, \qquad \text{etc.}$$

ii. (3 points) Find the matrix B for orthogonal projection onto W. In other words, the matrix B so that $Bx = x_W$ for every x in $\mathbf{R}^2$. Enter your answer below.

With $u = \begin{pmatrix}3\\1\end{pmatrix}$, we have

$$B = \frac{1}{u \cdot u} u u^T = \frac{1}{3^2 + 1^2}\begin{pmatrix}3\\1\end{pmatrix}\begin{pmatrix}3 & 1\end{pmatrix} = \frac{1}{10}\begin{pmatrix}9 & 3\\3 & 1\end{pmatrix}.$$

(b) Let $x = \begin{pmatrix}10\\-10\end{pmatrix}$.

(i) (3 points) Find x_W. In other words, find the orthogonal projection of x onto W. Fully simplify your answer.

$$x_W = Bx = \frac{1}{10}\begin{pmatrix}9 & 3\\3 & 1\end{pmatrix}\begin{pmatrix}10\\-10\end{pmatrix} = \frac{1}{10}\begin{pmatrix}60\\20\end{pmatrix} = \begin{pmatrix}6\\2\end{pmatrix}.$$

Alternatively, we could compute

$$x_W = \frac{u \cdot x}{u \cdot u} u = \frac{20}{10}\begin{pmatrix}3\\1\end{pmatrix} = \begin{pmatrix}6\\2\end{pmatrix}.$$

(ii) (2 points) Find $x_{W^\perp}$ and write your answer in the space below.

$$x_{W^\perp} = x - x_W = \begin{pmatrix}10\\-10\end{pmatrix} - \begin{pmatrix}6\\2\end{pmatrix} = \begin{pmatrix}4\\-12\end{pmatrix}.$$

10. Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,6), \qquad (2,4), \qquad (4,8).$$

Enter your answer below:

$$y = \_\_\_\_\_\_\_\_ x \;+\; \_\_\_\_\_\_\_\_ .$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

No line goes through all three points. The corresponding (inconsistent) system is

$$6 = M(0) + B$$

$$4 = M(2) + B$$

$$8 = M(4) + B$$

and the corresponding matrix equation is $Ax = b$ where $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$.

We solve $A^T A\widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 20 & 6 \\ 6 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 0 & 2 & 4 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix} = \begin{pmatrix} 40 \\ 18 \end{pmatrix}.$$

$$\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 20 & 6 & 40 \\ 6 & 3 & 18 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/6]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1/2 & 3 \\ 20 & 6 & 40 \end{array}\right) \xrightarrow{R_2 = R_2 - 20R_1} \left(\begin{array}{cc|c} 1 & 1/2 & 3 \\ 0 & -4 & -20 \end{array}\right)$$

$$\xrightarrow{R_2 = -R_2/4} \left(\begin{array}{cc|c} 1 & 1/2 & 3 \\ 0 & 1 & 5 \end{array}\right) \xrightarrow{R_1 = R_1 - (R_2/2)} \left(\begin{array}{cc|c} 1 & 0 & 1/2 \\ 0 & 1 & 5 \end{array}\right).$$

Thus $\widehat{x} = \begin{pmatrix} 1/2 \\ 5 \end{pmatrix}$. The line is

$$y = \frac{1}{2}x + 5.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.