Math 1553 Exam 2, SOLUTIONS, Spring 2025

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A+HP, 8:25-9:15 AM) Jankowski (C, 9:30-10:20 AM)

Al Ahmadieh (I, 2:00-2:50 PM) Al Ahmadieh (M, 3:30-4:20 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam.* If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, March 5.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If $\{v_1, v_2, v_3, v_4\}$ is a linearly dependent set of vectors in $\mathbf{R}^n$, then v_4 must be in $\text{Span}\{v_1, v_2, v_3\}$.

   ○ True

   ● False

   (b) If A is a 4×3 matrix and the columns of A are linearly independent, then $\dim(\text{Col } A) = 3$.

   ● True

   ○ False

   (c) Suppose A is a 5×4 matrix and b is a vector so that $Ax = b$ is consistent. Then the set of solutions to $Ax = b$ must be a subspace of $\mathbf{R}^4$.

   ○ True

   ● False

   (d) Suppose A is an $m \times n$ matrix with corresponding matrix transformation $T(x) = Ax$. If the RREF of A has a row of zeros, then T cannot be one-to-one.

   ○ True

   ● False

   (e) Suppose A is an $n \times n$ matrix and there is a vector b in $\mathbf{R}^n$ so that the equation $Ax = b$ has exactly one solution. Then A must be invertible.

   ● True

   ○ False

Solution: Problem 1.

(a) False. For example, the set of vectors below is linearly dependent, but v_4 is not in $\text{Span}\{v_1, v_2, v_3\}$.

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \quad v_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

(b) True: A has a pivot in each of its 3 columns since they are linearly independent, and the pivot columns form a basis for $\text{Col}(A)$, so $\dim(\text{Col } A) = 3$.

(c) False: if $b \neq 0$ then the solution set to $Ax = b$ can never be a subspace, because it does not contain the zero vector. As one specific example, take

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

Clearly, $x = 0$ is not a solution to $Ax = b$.

(d) False: for example, take $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then A has a row of zeros but T is one-to-one.

(e) True: $Ax = b$ having one solution means that A has a pivot in every column, therefore A is invertible by the Invertible Matrix Theorem.

2. **Full solutions are on the next page.**

(a) (3 points) Suppose v_1, v_2, and v_3 are vectors in $\mathbf{R}^n$. Which of the following statements are true? Fill in the bubble for all that apply.

● If $\{v_1, v_2, v_3\}$ is linearly independent, then the only solution to the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ is: $x_1 = 0$, $x_2 = 0$, $x_3 = 0$.

● If $\{v_1, v_2, v_3\}$ is linearly independent and b is a vector with the property that the vector equation $x_1v_1+x_2v_2+x_3v_3 = b$ is consistent, then the vector equation must have exactly one solution.

○ If $\{v_1, v_2, v_3\}$ is linearly dependent, then $x_1v_1+x_2v_2+x_3v_3 = 0$ for some scalars satisfying $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$.

(b) (2 pts) Find all values of h so that the following vectors form a basis for $\mathbf{R}^3$:

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 2 \\ 2 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 1 \\ -2 \\ h \end{pmatrix}.$$

○ $h = 0$ only ○ $h = -1$ only ○ $h = 1$ only ○ $h = -2$ only

○ all h except 0 ● all h except -1 ○ none of these

(c) (2 points) Suppose A is a 40×20 matrix and the solution set to $Ax = 0$ is 5-dimensional. Which **one** of the following describes the column space of A? Fill in the bubble for your answer.

○ Col(A) is a 15-dimensional subspace of $\mathbf{R}^{20}$.

● Col(A) is a 15-dimensional subspace of $\mathbf{R}^{40}$.

○ Col(A) is a 5-dimensional subspace of $\mathbf{R}^{20}$.

○ Col(A) is a 5-dimensional subspace of $\mathbf{R}^{40}$.

(d) (3 points) Suppose A is a 2×2 matrix, and let V be the set of all vectors x in $\mathbf{R}^2$ that satisfy $Ax = 3x$. Which of the following statements are true? Fill in the bubble for all that apply.

● V contains the zero vector.

● V closed under addition? In other words, if u and v are vectors in V, must it be true that $u + v$ is in V?

● V closed under scalar multiplication? In other words, if u is a vector in V and c is a scalar, must it be true that cu is in V?

Solution: Problem 2

(a) (i) is true: this is almost word-for-word the definition of linear independence.

(ii) is true: by the relationships between solution sets, the consistent equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ has a unique solution since $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has a unique solution (by linear independence).
Alternatively, we could reason that since $x_1v_1+x_2v_2+x_3v_3 = 0$ has exactly one solution due to linear independence, it follows that any consistent system's augmented matrix $\left(\begin{array}{ccc|c} v_1 & v_2 & v_3 & b \end{array}\right)$ will have pivot in every column except the rightmost column and therefore the system have exactly one solution.

(iii) is false: if the vectors are linearly dependent, then $x_1v_1 + x_2v_2 + x_3v_3 = 0$ where **at least one** of the scalars is nonzero. They don't all need to be nonzero. For example, take $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. The set is linearly dependent, but the only way to get $x_1v_1 + x_2v_2 + x_3v_3 = 0$ is to choose $x_1 = 0$ and $x_2 = 0$ (and x_3 any number you want).

(b) We row-reduce:

$$\begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & 2 & -2 \\ 1 & 2 & h \end{pmatrix} \xrightarrow{R_3=R_3-R_1} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 2 & h-1 \end{pmatrix} \xrightarrow{R_3=R_3-R_2} \begin{pmatrix} \boxed{1} & 0 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & h+1 \end{pmatrix}.$$

This matrix has a pivot and every row and column (which is to say, the three vectors are a basis for $\mathbf{R}^3$) precisely when h is NOT -1.

(c) $\text{Col}(A)$ lives in $\mathbf{R}^{40}$ since A has 40 rows, and by the Rank Theorem:

$$\dim(\text{Col } A)+\dim(\text{Nul } A) = 20, \quad \dim(\text{Col } A)+5 = 20, \quad \dim(\text{Col}(A)) = 15.$$

(d) (i) Yes, V contains the zero vector because $A(0) = 3(0) = 0$.
(ii) Yes, V is closed under addition. If $Au = 3u$ and $Av = 3v$, then

$$A(u+v) = Au + Av = 3u + 3v = 3(u+v), \quad \text{so } u+v \quad \text{is in } V.$$

(iii) Yes, V is closed under scalar multiplication: if $Au = 3u$ and c is a scalar, then $A(cu) = cAu = c(3u) = 3cu$, so cu is in V.

3. On this page, you do not need to show work and there is no partial credit. Parts (a) through (d) are unrelated.

(a) (2 points) Which **one** of the following linear transformations is onto? Clearly fill in the bubble for your answer.

○ $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x, y) = (x, x - y, 2x + y)$.

○ $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by $T(x, y, z) = (x - y - z,\ 3x - 3y - 3z)$.

● $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x, y) = (x - y,\ 2x - y)$.

○ $T : \mathbf{R}^3 \to \mathbf{R}^3$ that projects vectors onto the xy-plane.

(b) (2 points) Let A be an $m \times n$ matrix, and let T be the corresponding matrix transformation $T(x) = Ax$. Which **one** of the following statements guarantees that T is one-to-one?

○ For each x in $\mathbf{R}^n$, there is at least one y in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each x in $\mathbf{R}^n$, there is at most one y in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each y in $\mathbf{R}^m$, there is at least one x in $\mathbf{R}^n$ so that $T(x) = y$.

● For each y in $\mathbf{R}^m$, there is at most one x in $\mathbf{R}^n$ so that $T(x) = y$.

(c) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by $90°$ **clockwise**. Solve $T(x) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. Fill in the bubble for your answer below.

○ $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ ● $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} -3 \\ -1 \end{pmatrix}$ ○ none of these

(d) (4 points) Suppose A is a 7×9 matrix and B is a 9×4 matrix, and let T be the matrix transformation $T(x) = ABx$.

(i) What is the domain of T? ● $\mathbf{R}^4$ ○ $\mathbf{R}^7$ ○ $\mathbf{R}^9$

(ii) What is the codomain of T? ○ $\mathbf{R}^4$ ● $\mathbf{R}^7$ ○ $\mathbf{R}^9$

(iii) Which of the following statements are true? Fill in the bubble for all that apply.

○ If x is in the domain of T, then ABx must be in the column space of B.

● Every vector in the null space of B must also be in the null space of AB.

Solution: Problem 3.

(a) (i) We don't even need to write the corresponding matrix: it will be 3×2 and therefore cannot have a pivot in every row, so T cannot possibly be onto.

(ii) $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) \end{pmatrix} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -3 & -3 \end{pmatrix}$ which only has one pivot, therefore T is not onto.

(iii) $A = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix} \xrightarrow{R_2 = R_2 - 2R_1} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ which has a pivot in every row, so T is onto.

(iv) It is a standard fact that T is not onto, since its range is only the xy-plane of $\mathbf{R}^3$ rather than all of $\mathbf{R}^3$: Note $T(x, y, z) = (x, y, 0)$, so for example $(0, 0, 1)$ is not in the range of T.

(b) (i) and (ii) are each almost the definition of a transformation, and neither says anything about whether T is one-to-one.

(iii) is equivalent to the statement that T is onto, while (iv) means that T is one-to-one.

(c) We need to solve $T(x) = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. The matrix for T is $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, so we solve:

$$\left(\begin{array}{cc|c} 0 & 1 & 3 \\ -1 & 0 & 1 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 3 \end{array}\right), \qquad \text{so } x = \begin{pmatrix} -1 \\ 3 \end{pmatrix}.$$

We could also have done it geometrically: x is the vector that, when rotated $90°$ clockwise, will equal $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

This means that x is "$\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ rotated $90°$ **counterclockwise**" which is $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$.

(d) (i) and (ii): AB is 7×4, so T has domain $\mathbf{R}^4$ and codomain $\mathbf{R}^7$.

(iii): ABx is not in the column space of B. In fact, the column space of B is a subspace of $\mathbf{R}^9$, but ABx is a vector in $\mathbf{R}^7$.

If x is in the null space of B (which means $Bx = 0$), then it is in the null space of AB, since $ABx = A(0) = 0$.

4. (a) (2 points) Suppose A is an invertible $n \times n$ matrix. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

● The RREF of A is the $n \times n$ identity matrix.

○ If B is an invertible $n \times n$ matrix, then $(AB)^{-1} = A^{-1}B^{-1}$.

(b) (2 points) Let $A = \begin{pmatrix} 5 & 3 \\ 2 & 1 \end{pmatrix}$. Find $A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Fill in the bubble for your answer.

○ $\begin{pmatrix} -1 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ ● $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} -1 \\ 3 \end{pmatrix}$

○ $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} -5 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} -5 \\ -2 \end{pmatrix}$

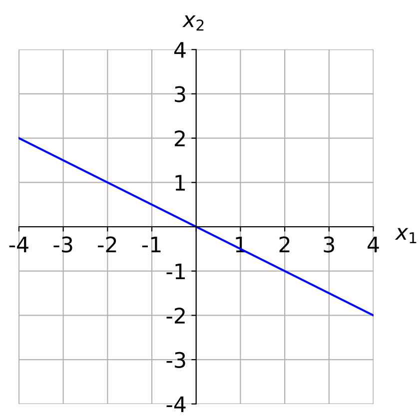
(c) (2 pts) Write one specific subspace V of $\mathbf{R}^3$ that contains $\begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}$.

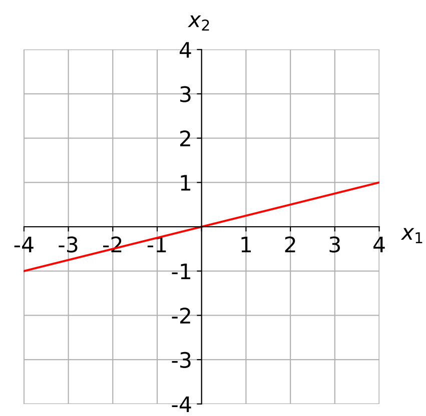
Solution is on next page

(d) (4 points) Let $A = \begin{pmatrix} 2 & -8 \\ -1 & 4 \end{pmatrix}$.

On the **left** graph below, draw Col(A) carefully.

On the **right** graph below, draw Nul(A) carefully.





Solution: Problem 4.

(a) (i) Yes, this is straight from the Invertible Matrix Theorem. The RREF of A is the $n \times n$ identity matrix if and only if it has n pivots, which means A is invertible.

(ii) No: the correct formula is $(AB)^{-1} = B^{-1}A^{-1}$.

(b) We can do this multiple ways. One way is to compute

$$A^{-1} = \frac{1}{5(1) - 2(3)} \begin{pmatrix} 1 & -3 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 2 & -5 \end{pmatrix}, \quad \text{and} \quad A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

Alternative method: $A^{-1}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the vector x so that $Ax = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so we can solve:

$$\left(\begin{array}{cc|c} 5 & 3 & 1 \\ 2 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 2 \end{array}\right) \quad \text{so} \quad x = \begin{pmatrix} -1 \\ 2 \end{pmatrix}.$$

(c) There are many answers possible. For example, $\text{Span}\left\{\begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}\right\}$ is a subspace of $\mathbf{R}^3$ containing $\begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}$. Another correct answer is $\mathbf{R}^3$ itself.

One **incorrect** answer is the single vector $\begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}$, or in set notation $\left\{\begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}\right\}$.

(d) $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} -8 \\ 4 \end{pmatrix}\right\} = \text{Span}\left\{\begin{pmatrix} 2 \\ -1 \end{pmatrix}\right\}$.

For $\text{Nul}(A)$:

$$A = \left(\begin{array}{cc|c} 2 & -8 & 0 \\ -1 & 4 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right),$$

so the null space of A is the line $x_1 = 4x_2$, which is $\text{Span}\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}$.

5. Free response. Show your work! A correct answer without appropriate work will receive little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the linear transformation given by $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+y \\ x \\ y-3x \end{pmatrix}$.

Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by 90° **counter**clockwise.

(a) (3 points) Find the standard matrix A for T and write it in the space below. Show your work.

$$A = \left(T\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -3 & 1 \end{pmatrix}$$

(b) (2 points) Write the standard matrix B for U in the space below. Evaluate all trigonometric functions you write. Do not leave your answers in terms of sine and cosine.

$$B = \left(U\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad U\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} \cos(90^\circ) & -\sin(90^\circ) \\ \sin(90^\circ) & \cos(90^\circ) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(c) (1 point) Which composition makes sense: $T \circ U$ or $U \circ T$? Fill in the correct bubble below. You do not need to show your work on this part.

● $T \circ U$ ○ $U \circ T$

The composition $T \circ U$ makes sense: if x is in $\mathbf{R}^2$ then $U(x)$ is a vector in $\mathbf{R}^2$, so it is in the domain of T and we can take $T(U(x))$.

$U \circ T$ does not make sense: the domain of U is $\mathbf{R}^2$ but the range of T lives in $\mathbf{R}^3$, so it is not possible to take $U(T(x))$ for any x in $\mathbf{R}^2$.

(d) (4 points) Compute the standard matrix C for the composition you circled in part (c). Enter your answer below.
The matrix for $T \circ U$ is AB, so

$$C = AB = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & -1 \\ 1 & 3 \end{pmatrix}$$

6. Free response. Show your work on (b) and (d), where a correct answer without sufficient work may receive little or no credit. You not need to show your work on (a) or (c). For this page, consider the matrix A and its reduced row echelon form given below.

$$A = \begin{pmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & -1 & -3 & 1 & 3 \\ -2 & -1 & 1 & -1 & 3 \\ 0 & 1 & 3 & 0 & -4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 3 & 0 & -4 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (2 points) Write a basis for Col A. (no work required on this part)
Solution: The pivot columns of A form a basis for Col(A):

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ -1 \\ 0 \end{pmatrix} \right\}.$$

However, it turns out that in this problem, any choice of three columns of A will be a basis for the column space of A.

(b) (4 points) Find a basis for Nul A.
Solution: From the RREF of A, the solution set to $Ax = 0$ is given by $x_1 - 2x_3 + x_5 = 0$, $x_2 + 3x_3 - 4x_5 = 0$, and $x_4 - x_5 = 0$, where x_3 and x_5 are free.

$$x_1 = 2x_3 - x_5, \quad x_2 = -3x_3 + 4x_5, \quad x_3 = x_3 \text{ (real)}, \quad x_4 = x_5, \quad x_5 = x_5 \text{ (real)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_3 - x_5 \\ -3x_3 + 4x_5 \\ x_3 \\ x_5 \\ x_5 \end{pmatrix} = \begin{pmatrix} 2x_3 \\ -3x_3 \\ x_3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_5 \\ 4x_5 \\ 0 \\ x_5 \\ x_5 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_5 \begin{pmatrix} -1 \\ 4 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

The two vectors in red form a basis for Nul(A).

(c) (2 points) Define a linear transformation $T : \mathbf{R}^5 \to \mathbf{R}^4$ by $T(x) = Ax$.

- Is T one-to-one? ○ Yes ● No
- Is T onto? ○ Yes ● No

(d) (2 points) Is $\begin{pmatrix} 0 \\ -3 \\ -3 \\ 3 \end{pmatrix}$ in the span of the columns of A? Justify your answer.

Yes. In fact, we can see this without doing any work. This vector is 3 times the second column of A, so $A \begin{pmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ -3 \\ 3 \end{pmatrix}$.

7. Free response. Show your work except on part (c)! A correct answer without sufficient work in (a) or (b) will receive little or no credit. Parts (a) through (c) are unrelated.

(a) (3 points) Suppose $T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ is a linear transformation satisfying:

$$T\begin{pmatrix}0\\1\\-2\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix} \quad \text{and} \quad T\begin{pmatrix}1\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

Find $T\begin{pmatrix}-2\\3\\-8\end{pmatrix}$. Hint: $\begin{pmatrix}-2\\3\\-8\end{pmatrix} = \begin{pmatrix}0\\1\\-2\end{pmatrix} - 2\begin{pmatrix}1\\-1\\3\end{pmatrix}$.

Solution: We use the properties of linearity.

$$T\begin{pmatrix}-2\\3\\-8\end{pmatrix} = T\left(\begin{pmatrix}0\\1\\-2\end{pmatrix} - 2\begin{pmatrix}1\\-1\\3\end{pmatrix}\right) = T\begin{pmatrix}0\\1\\-2\end{pmatrix} - 2T\begin{pmatrix}1\\-1\\3\end{pmatrix}$$

$$= \begin{pmatrix}0\\1\end{pmatrix} - 2\begin{pmatrix}2\\-1\end{pmatrix} = \begin{pmatrix}-4\\3\end{pmatrix}.$$

(b) (4 points) Let $A = \begin{pmatrix}x & y\\4 & -2\end{pmatrix}$. Find all values of x and y so that $A^2 = A$.

Enter your answer here: $x = 3$ and $y = -3/2$

Solution: We compute $A^2 = \begin{pmatrix}x & y\\4 & -2\end{pmatrix}\begin{pmatrix}x & y\\4 & -2\end{pmatrix} = \begin{pmatrix}x^2+4y & xy-2y\\4x-8 & 4y+4\end{pmatrix}$.

Setting $A^2 = A$ gives us the following for the second row:

$$4x - 8 = 4 \implies 4x = 12 \implies x = 3.$$

$$4y + 4 = -2 \implies 4y = -6 \implies y = -3/2.$$

We could also check that

$$x^2 + 4y = 9 - 6 = 3 = x \quad \text{and} \quad xy - 2y = -\frac{9}{2} - 2\left(\frac{-3}{2}\right) = -\frac{3}{2} = y,$$

so the first row of A^2 equals the first row of A.

(c) (3 points) Write a single matrix A whose corresponding linear transformation T (given by $T(x) = Ax$) satisfies both of the following conditions.

- The solution set to $T(x) = 0$ is a line.
- The range of T is the plane in $\mathbf{R}^3$ spanned by $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$.

$$A = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}.$$

Solution: There are infinitely many correct answers.

- From the first condition, there must be one free variable in the solution set to $Ax = 0$, so exactly one column in A must fail to have a pivot.
- From the second condition, the columns of A must span the plane in $\mathbf{R}^3$ spanned by $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix}$, so A has exactly two pivot columns.

This means that A must have 3 columns total, and that 2 of them must be the appropriate pivot columns.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 4 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 4 & 3 \\ 0 & 1 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 4 & -5 \\ 0 & 1 & -1 \end{pmatrix}, \quad \text{etc.}$$

In each case above, we put the two pivot columns first, then created a third column that was a linear combination of the first two columns.

There are plenty more examples, such as:

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}.$$

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