Supplemental problems: §5.1

1. True or false. Answer true if the statement is always true. Otherwise, answer false.

a) If A and B are $n \times n$ matrices and A is row equivalent to B, then A and B have the same eigenvalues.

b) If A is an $n \times n$ matrix and its eigenvectors form a basis for $\mathbf{R}^n$, then A is invertible.

c) If 0 is an eigenvalue of the $n \times n$ matrix A, then $\text{rank}(A) < n$.

d) The diagonal entries of an $n \times n$ matrix A are its eigenvalues.

e) If A is invertible and 2 is an eigenvalue of A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

f) If $\det(A) = 0$, then 0 is an eigenvalue of A.

g) If v and w are eigenvectors of a square matrix A, then so is $v + w$.

Solution.

a) False. For instance, the matrices $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ are row equivalent, but have different eigenvalues.

b) False. For example, the zero matrix is not invertible but its eigenvectors form a basis for $\mathbf{R}^n$.

c) True. If $\lambda = 0$ is an eigenvalue of A then A is not invertible so its associated transformation $T(x) = Ax$ is not onto, hence $\text{rank}(A) < n$.

d) False. This is true if A is triangular, but not in general.
For example, if $A = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$ then the diagonal entries are 2 and 0 but the only eigenvalue is $\lambda = 1$, since solving the characteristic equation gives us

$$(2-\lambda)(-\lambda) - (1)(-1) = 0 \qquad \lambda^2 - 2\lambda + 1 = 0 \qquad (\lambda - 1)^2 = 0 \qquad \lambda = 1.$$

e) True. Let v be an eigenvector corresponding to the eigenvalue 2.

$$Av = 2v \implies A^{-1}Av = A^{-1}(2v) \implies v = 2A^{-1}v \implies \frac{1}{2}v = A^{-1}v.$$

Therefore, v is an eigenvector of A^{-1} corresponding to the eigenvalue $\frac{1}{2}$.

f) True. If $\det(A) = 0$ then A is not invertible, so $Av = 0v$ has a nontrivial solution.

g) False. Take $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Note e_1 is an eigenvector corresponding to $\lambda = 1$ and e_2 is an eigenvector corresponding to $\lambda = 2$, but $e_1 + e_2$ is not an eigenvalue of A.

2. In this problem, you need not explain your answers; just circle the correct one(s).

Let A be an $n \times n$ matrix.

a) Which **one** of the following statements is correct?

1. An eigenvector of A is a vector v such that $Av = \lambda v$ for a nonzero scalar λ.
2. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a scalar λ.
3. An eigenvector of A is a nonzero scalar λ such that $Av = \lambda v$ for some vector v.
4. An eigenvector of A is a nonzero vector v such that $Av = \lambda v$ for a nonzero scalar λ.

b) Which **one** of the following statements is **not** correct?

1. An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.
2. An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.
3. An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.
4. An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

Solution.

a) Statement 2 is correct: an eigenvector must be nonzero, but its eigenvalue may be zero.

b) Statement 2 is incorrect: the solution v must be nontrivial.

3. Find a basis $\mathcal{B}$ for the (-1)-eigenspace of $Z = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$

Solution.

For $\lambda = -1$, we find $\mathrm{Nul}(Z - \lambda I)$.

$$\left(Z - \lambda I \mid 0 \right) = \left(Z + I \mid 0 \right) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Therefore, $x = -y$, $y = y$, and $z = 0$, so

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y \\ y \\ 0 \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

A basis is $\mathcal{B} = \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$. We can check to ensure $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector with corresponding eigenvalue -1:

$$Z \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2+3 \\ -3+2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

4. Suppose A is an $n \times n$ matrix satisfying $A^2 = 0$. Find all eigenvalues of A. Justify your answer.

Solution.

If λ is an eigenvalue of A and $v \neq 0$ is a corresponding eigenvector, then

$$Av = \lambda v \implies A(Av) = A\lambda v \implies A^2 v = \lambda(Av) \implies 0 = \lambda(\lambda v) \implies 0 = \lambda^2 v.$$

Since $v \neq 0$ this means $\lambda^2 = 0$, so $\lambda = 0$. This shows that 0 is the only possible eigenvalue of A.

On the other hand, $\det(A) = 0$ since $(\det(A))^2 = \det(A^2) = \det(0) = 0$, so 0 must be an eigenvalue of A. Therefore, the only eigenvalue of A is 0.

5. Match the statements (i)-(v) with the corresponding statements (a)-(e). All matrices are 3×3. There is a unique correspondence. Justify the correspondences in words.

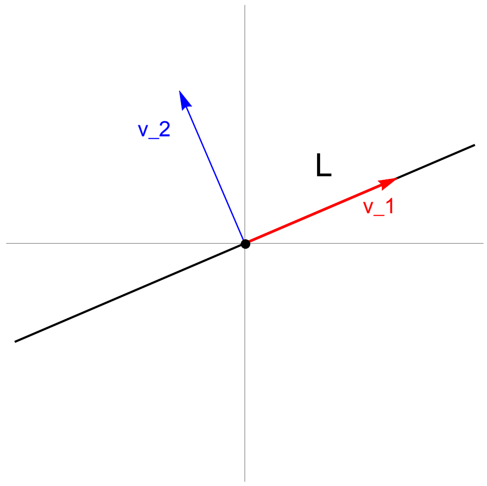
(i) $Ax = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix}$ has a unique solution.
(ii) The transformation $T(v) = Av$ fixes a nonzero vector.
(iii) A is obtained from B by subtracting the third row of B from the first row of B.
(iv) The columns of A and B are the same; except that the first, second and third columns of A are respectively the first, third, and second columns of B.
(v) The columns of A, when added, give the zero vector.

(a) 0 is an eigenvalue of A.
(b) A is invertible.
(c) $\det(A) = \det(B)$
(d) $\det(A) = -\det(B)$
(e) 1 is an eigenvalue of A.

Solution.

(i) matches with (b).
(ii) matches with (e).
(iii) matches with (c).
(iv) matches with (d).
(v) matches with (a).

6. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation which reflects across the line L drawn below, and let A be the standard matrix for T.



a) Write all eigenvalues of A.

This problem is similar to an example in our class notes near the end of section 5.1.

$\lambda_1 = 1$ and $\lambda_2 = -1$.

(The equation of the line was not given, and it is irrelevant: A fixes every vector along the line L, while A flips every vector perpendicular to L.)

b) For each eigenvalue of A, draw one eigenvector on the graph above. Your eigenvector does not need to be perfect, but it should be reasonably accurate.

Above, v_1 corresponds to $\lambda_1 = 1$, while v_2 corresponds to $\lambda_2 = -1$.
Many answers are possible: v_1 can be any nonzero vector on L (going up-to-right or down-to-left), while v_2 can be any nonzero vector perpendicular to L (going up-to-left or down-to-right).

Both (a) and (b) can be done without any computations or algebra. You don't need to justify your answers, as long as your pictures in part (b) are clear.

Algebraically, the problem would have been much more painful!

The line is actually the line $y = \frac{3}{7}x$, the matrix is $A = \frac{1}{29}\begin{pmatrix} 20 & 21 \\ 21 & -20 \end{pmatrix}$, and the eigenvectors drawn are $v_1 = \begin{pmatrix} 7/3 \\ 1 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -1 \\ 7/3 \end{pmatrix}$.