Math 1553 Worksheet §3.4

Solutions

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

a) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

b) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

Solution.

a) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, then it has n pivots. This is the same as saying that the transformation $T(x) = Ax$ is both one-to-one and onto. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

b) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

2. A is $m \times n$ matrix, B is $n \times m$ matrix. Select all correct answers from the box. It is possible to have more than one correct answer.

a) Suppose x is in $\mathbf{R}^m$. Then ABx *must be* in:

$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

b) Suppose x in $\mathbf{R}^n$. Then BAx *must be* in:

$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

c) If $m > n$, then columns of AB could be linearly $\boxed{independent, \quad dependent}$

d) If $m > n$, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

Solution.

Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

a) $\boxed{\text{Col}(A)}$. Denote $w := Bx$, which is a vector in $\mathbf{R}^n$. $ABx = A(Bx)$ is multiplying A with w which will end up with "linear combination of columns of A". So ABx is in Col(A).

b) $\boxed{\text{Col}(B)}$. Similarly, $BAx = B(Ax)$ is multiplying B with Ax, a vector in R^m, which will end up with "linear combination of columns of B". So BAx is in Col(B).

c) $\boxed{dependent}$. Since $m > n$ means A matrix can have at most n pivots. So $dim(\text{Col}(A)) \leq n$. Notice from first question we know $\text{Col}(AB) \subset \text{Col}(A)$ which has dimension at most n. That means AB can have at most n pivots. But AB is $m \times m$ matrix, then columns of AB must be dependent.

d) $\boxed{independent, dependent}$. Both are possible. Since $m > n$ means B matrix can have at most n pivots. then $\text{Col}(BA) \subset \text{Col}(B)$ means BA can have at most n pivots. Since BA is $n \times n$ matrix, then the columns of BA will be linearly independent when there are n pivots or linearly dependent when there are less than n pivots. Here are two examples.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

To summarize what we are actually study here, there are several relations between these subspaces. The symbol $\subseteq$ means "is contained in (or possibly equal to)..."

$$\begin{aligned} \text{Col}(AB) &\subseteq \text{Col}(A); \\ \text{Col}(BA) &\subseteq \text{Col}(B); \\ \text{Nul}(A) &\subseteq \text{Nul}(BA); \\ \text{Nul}(B) &\subseteq \text{Nul}(AB); \end{aligned}$$

3. Consider the following linear transformations:

$T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$ T projects onto the xy-plane, forgetting the z-coordinate

$U : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ U rotates clockwise by $90°$

$V : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ V scales the x-direction by a factor of 2.

Let A, B, C be the matrices for T, U, V, respectively.

a) Write A, B, and C.

b) Compute the matrix for $U \circ V \circ T$.

Solution.

a) We plug in the unit coordinate vectors:

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$U(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

$$V(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad V(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

b) $BCA = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$.