Math 1553 Worksheet §§5.6-6.3

1. Courage Soda and Dexter Soda compete for a market of 210 customers who drink soda each day.
Today, Courage has 80 customers and Dexter has 130 customers. Each day:
70% of Courage Soda's customers keep drinking Courage Soda, while 30% switch to Dexter Soda.
40% of Dexter Soda's customers keep drinking Dexter Soda, while 60% switch to Courage Soda.

a) Write a stochastic matrix A and a vector x so that Ax will give the number of customers for Courage Soda and Dexter Soda (in that order) tomorrow.
You do not need to compute Ax.

$$A = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 80 \\ 130 \end{pmatrix}.$$

b) Find the steady-state vector for A.

$$\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} -0.3 & 0.6 & 0 \\ 0.3 & -0.6 & 0 \end{array}\right) \xrightarrow[R_1=R_1/(-0.3)]{R_2=R_2+R_1} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so $x_1 = 2x_2$ and x_2 is free. A 1-eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so the steady state vector is $w = \dfrac{1}{2+1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

c) Use your answer from (b) to determine the following: in the long run, roughly how many daily customers will Courage Soda have?

As n gets large, $A^n \begin{pmatrix} 80 \\ 130 \end{pmatrix}$ approaches $210 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 140 \\ 70 \end{pmatrix}$. Courage will have roughly 140 customers.

2. Let W be the set of all vectors in $\mathbf{R}^3$ of the form $(x, x-y, y)$ where x and y are real numbers.

a) Find a basis for $W^\perp$.

b) Let $x = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$. Find the projection x_W of x onto the subspace W and the orthogonal projection $x_{W^\perp}$ of x onto the subspace $W^\perp$.

Solution.

a) A vector in W has the form

$$\begin{pmatrix} x \\ x-y \\ y \end{pmatrix} = \begin{pmatrix} x \\ x \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -y \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \qquad \text{so } W \text{ has basis } \left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}.$$

To get $W^\perp$ we find $\mathrm{Nul}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$ which gives us

$$x_1 = -x_3, \quad x_2 = x_3, \quad x_3 = x_3 \text{ (free)},$$

so $W^\perp$ has basis $\left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\}$.

b) Let A be the matrix whose columns are the basis vectors for W: $A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$.

We calculate $A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

and $A^T x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$, and solve $A^T A v = A^T x$.

$$\left(A^T A \mid A^T x\right) = \left(\begin{array}{cc|c} 2 & -1 & 0 \\ -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \end{array}\right) \implies v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Then final answer for x_W is $x_W = Av = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$,

and $x_{W^\perp} = x - x_W = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$.