

Math 1553 Worksheet: Chapter 5.1-5.2

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.
- a) If v_1 and v_2 are linearly independent eigenvectors of an $n \times n$ matrix A , then they must correspond to different eigenvalues.
 - b) If A is a 3×3 matrix with characteristic polynomial $-\lambda(\lambda - 5)^2$, then the 5-eigenspace is 2-dimensional.

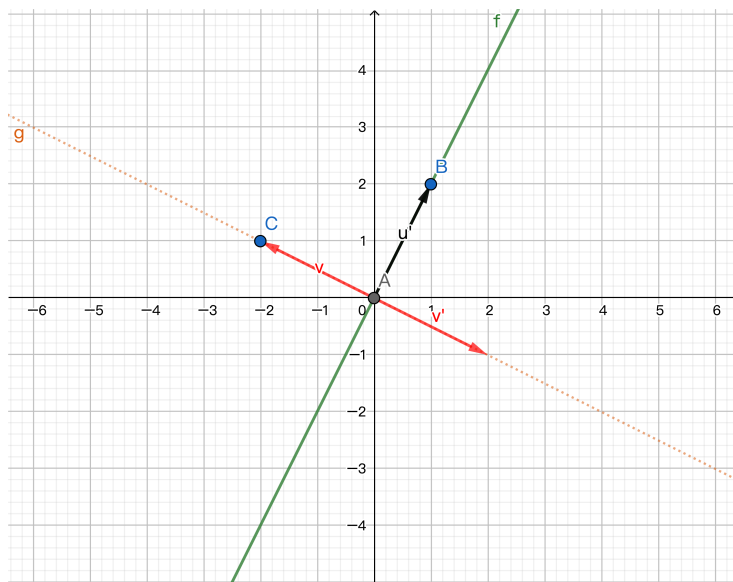
Solution.

- a) False. For example, if $A = I_2$ then $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ are linearly independent eigenvectors both corresponding to the eigenvalue $\lambda = 1$.
- b) False. The dimension of the 5-eigenspace can be 1 or 2. For example, the matrix $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 2-dimensional, whereas the matrix $\begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ has a 5-eigenspace which is 1-dimensional. Both matrices have characteristic polynomial $-\lambda(5 - \lambda)^2$.

2. Let A be the matrix of the linear transformation $T : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ that reflects vectors over the line $y = 2x$ in $\mathbf{R}^2$. Find the eigenvectors and eigenvalues of A without doing any matrix calculations. (Draw a picture!)

Solution.

Here is a picture:



T fixes every vector along the line $y = 2x$, so $\lambda = 1$ is an eigenvalue and its eigenvectors are all vectors $\begin{pmatrix} t \\ 2t \end{pmatrix}$ where $t \neq 0$.

T flips every vector along the line perpendicular to $y = 2x$, which is $y = -\frac{1}{2}x$ (for example, $T(-2, 1) = (2, -1)$). Therefore, $\lambda = -1$ is an eigenvalue and its eigenvectors are all vectors of the form $\begin{pmatrix} s \\ -\frac{1}{2}s \end{pmatrix}$ where $s \neq 0$.

3. Find the eigenvalues and a basis for each eigenspace of $A = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 2 & 4 \\ 0 & 0 & -1 \end{pmatrix}$.

Solution.

We solve $0 = \det(A - \lambda I)$.

$$\begin{aligned} 0 &= \det \begin{pmatrix} 2-\lambda & 3 & 1 \\ 3 & 2-\lambda & 4 \\ 0 & 0 & -1-\lambda \end{pmatrix} = (-1-\lambda)(-1)^6 \det \begin{pmatrix} 2-\lambda & 3 \\ 3 & 2-\lambda \end{pmatrix} = (-1-\lambda)((2-\lambda)^2 - 9) \\ &= (-1-\lambda)(\lambda^2 - 4\lambda - 5) = -(\lambda + 1)^2(\lambda - 5). \end{aligned}$$

So $\lambda = -1$ and $\lambda = 5$ are the eigenvalues.

$$\underline{\lambda = -1}: (A + I \mid 0) = \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 3 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[\text{then } R_2=R_2/3]{R_2=R_2-R_1} \left(\begin{array}{ccc|c} 3 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) \xrightarrow[\text{then } R_1=R_1/3]{R_1=R_1-R_2} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

with solution $x_1 = -x_2$, $x_2 = x_2$, $x_3 = 0$. The (-1) -eigenspace has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$\lambda = 5$:

$$(A - 5I \mid 0) = \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 3 & -3 & 4 & 0 \\ 0 & 0 & -6 & 0 \end{array} \right) \xrightarrow[\substack{R_2=R_2+R_1 \\ R_3=R_3/(-6)}]{R_2=R_2+R_1} \left(\begin{array}{ccc|c} -3 & 3 & 1 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right) \xrightarrow[\text{then } R_2 \leftrightarrow R_3, R_1/(-3)]{R_1=R_1-R_3, R_2=R_2-5R_3} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

with solution $x_1 = x_2$, $x_2 = x_2$, $x_3 = 0$. The 5-eigenspace has basis $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$.