Math 1553 Final, SOLUTIONS, Fall '25, Version B

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25 AM) Kim (B, 8:00 AM) Kim (C, 9:00 AM)

Callis (D, 10:00 AM) Short (E, 9:30 AM) Shi (F, 11:00 AM)

Short (H, 12:30 PM) He (I, 2:00 PM) Stokolosa (L, 3:30 PM)

Van Why (M, 3:30 PM) Yap (N, 5:00 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- Calculators and cell phones are not allowed. Aids of any kind (notes, text, etc.) are not allowed. If you use pen, you must use black ink. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero. Simplify all fractions and trig functions.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 8:50 PM on Tuesday, December 9.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) Let A be a 5×4 matrix whose null space is a line. Then its corresponding linear transformation $T(x) = Ax$ must be onto.

   ○ True

   ● False

   (b) If A is a 3×3 matrix and $A\begin{pmatrix}1\\0\\4\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$, then $\det(A) = 0$.

   ● True

   ○ False

   (c) Suppose A and B are $n \times n$ matrices with the same characteristic polynomial. If A is diagonalizable, then B must also be diagonalizable.

   ○ True

   ● False

   (d) Suppose W is a subspace of $\mathbf{R}^n$. Then there must be some matrix A so that $\mathrm{Col}(A) = W$ and $\mathrm{Nul}(A) = W^{\perp}$.

   ● True

   ○ False

   (e) Suppose $\hat{x}$ is a least squares solution to a matrix equation $Ax = b$. Then $\hat{x}$ is the closest vector to b in the column space of A.

   ○ True

   ● False

Problem 1 Solution.

(a) False: the null space of the 5×4 matrix A is the line, so there is 1 free variable in the solution set to $Ax = 0$. This means that the 4 columns of A contain only 3 pivots. Therefore, A has 2 rows without pivots, so T is not onto.

(b) True: we are given that $Ax = 0$ has a non-trivial solution. Therefore, A is not invertible, so $\det(A) = 0$.

(c) False. For example, the matrices $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ both have the characteristic polynomial λ^2. However, A is diagonalizable but B is not.

(d) True: the matrix for orthogonal projection onto W satisfies $\mathrm{Col}(A) = W$ and $\mathrm{Nul}(A) = W^{\perp}$.

(e) False: it is $A\hat{x}$ (*not* $\hat{x}$) that is the closest vector to b in the column space of A.

2. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (3 pts) Let V be the set of all vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ with the property that $xy \geq 0$. Which of the subspace properties does V satisfy? Fill in the bubble for all that apply.

● V contains the zero vector.

○ V closed under addition. In other words, if u and v are vectors in V, then $u + v$ must be in V.

● V closed under scalar multiplication. In other words, if u is a vector in V and c is a scalar, then cu must be in V.

(b) (3 points) Suppose A is a 80×100 matrix and the solution set to $Ax = 0$ has 25 free variables. Which of the following statements are true? Fill in the bubble for all that apply.

○ The null space of A is a 25-dimensional subspace of $\mathbf{R}^{80}$.

○ The column space of A is a 75-dimensional subspace of $\mathbf{R}^{100}$.

● $\dim(\text{Row } A) = 75$.

(c) (2 points) Let B be the matrix for orthogonal projection onto the subspace $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; 8x + y - 16z = 0 \right\}$. Which **one** of the following is true? Fill in the bubble for your answer.

● The vector $\begin{pmatrix} 1 \\ 8 \\ 1 \end{pmatrix}$ is in $\text{Col}(B)$.

○ The vector $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ is in $\text{Nul}(B)$.

○ $\dim(\text{Nul}(B - I)) = 1$.

○ If x is a vector in $\mathbf{R}^3$, then x is in W or $W^\perp$.

(d) (2 points) Find the area of the triangle with vertices $(-2, -3)$, $(3, 1)$, and $(5, 13)$. Fill in the bubble for your answer below.

○ 4 ○ 14 ○ 18 ● 26 ○ 28 ○ 32

○ 52 ○ 54 ○ 108 ○ none of these

Problem 2 Solution.

(a) (i) Yes, V contains the zero vector since $(0)(0) = 0 \geq 0$.

(ii) No, V is not closed under addition. For example, take $u = \begin{pmatrix}1\\0\end{pmatrix}$ and $v = \begin{pmatrix}0\\-1\end{pmatrix}$. Then u and v are both in V since $1 \cdot 0 = 0$ and $0 \cdot (-1) = 0$. However, $u + v = \begin{pmatrix}1\\-1\end{pmatrix}$, which is NOT in V since $1 \cdot (-1) < 0$.

(iii) Yes, V is closed under scalar multiplication. Suppose $u = \begin{pmatrix}x\\y\end{pmatrix}$ is in V and c is any scalar, so $xy \geq 0$. Then $cu = \begin{pmatrix}cx\\cy\end{pmatrix}$, which is in V since $(cx)(cy) = c^2(xy) \geq 0$.

(b) By the Rank Theorem we know

$$\dim(\mathrm{Col}(A)) + \dim(\mathrm{Nul}(A)) = 100.$$

We are given $\dim(\mathrm{Nul}(A)) = 25$, so $\dim(\mathrm{Col}(A)) = 75$.

(i) No, the null space is a subspace of $\mathbf{R}^{100}$.

(ii) No, the column space of A is a subspace of $\mathbf{R}^{80}$.

(iii) Yes, it is a fundamental fact that the row space and column space have the same dimension, so $\dim \mathrm{Row}(A) = 75$.

(c) (i) is true: we see $\begin{pmatrix}1\\8\\1\end{pmatrix}$ is in W since $8(1) + 8 - 16(1) = 0$, so $B\begin{pmatrix}1\\8\\1\end{pmatrix} = \begin{pmatrix}1\\8\\1\end{pmatrix}$.

(ii) is not true: note $\begin{pmatrix}2\\0\\1\end{pmatrix}$ is in W since $8(2) + 0 - 16(1) = 0$, so $B\begin{pmatrix}2\\0\\1\end{pmatrix} = \begin{pmatrix}2\\0\\1\end{pmatrix}$, therefore the vector is **not** in the null space of B.

(iii) is not true: the 1-eigenspace of B is W, which is 2-dimensional.

(iv) is not true: every x in $\mathbf{R}^2$ is a **linear combination** of a vector in W and a vector in $W^\perp$, but x is not necessarily **in** W or $W^\perp$.

(d) The vector from $(-2, -3)$ to $(3, 1)$ is $\begin{pmatrix}5\\4\end{pmatrix}$, and from $(-2, -3)$ to $(5, 13)$ is $\begin{pmatrix}7\\16\end{pmatrix}$. The area of the triangle is

$$\frac{1}{2}\left|\det\begin{pmatrix}5 & 7\\4 & 16\end{pmatrix}\right| = \frac{1}{2}(80 - 28) = \frac{52}{2} = 26.$$

3. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 points) Choose the **one** matrix A below with the property that

$$\text{Col }(A) = \text{Span}\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\} \qquad \text{and} \qquad \text{Nul }(A) = \text{Span}\left\{\begin{pmatrix}3\\-1\end{pmatrix}\right\}.$$

○ $A = \begin{pmatrix}-2 & 6\\1 & -3\end{pmatrix}$ ○ $A = \begin{pmatrix}-2 & 3\\1 & -1\end{pmatrix}$ ● $A = \begin{pmatrix}-2 & -6\\1 & 3\end{pmatrix}$

○ $A = \begin{pmatrix}-2 & -2/3\\1 & 1/3\end{pmatrix}$ ○ $A = \begin{pmatrix}-2 & 1\\1 & 3\end{pmatrix}$ ○ $A = \begin{pmatrix}-2 & 1\\3 & 1\end{pmatrix}$

(b) (2 points) Suppose $\det\begin{pmatrix}a & b & c\\d & e & f\\g & h & i\end{pmatrix} = 1$. Find $\det\begin{pmatrix}a & b & c\\3g & 3h & 3i\\5g-4d & 5h-4e & 5i-4f\end{pmatrix}$.

○ 3 ○ 4 ○ 5 ● 12 ○ 15 ○ 20

○ −3 ○ −4 ○ −5 ○ −12 ○ −15 ○ −20

○ 60 ○ −60 ○ none of these ○ not enough info

(c) Suppose A is a 3×3 matrix whose null space is the span of $\begin{pmatrix}1\\-4\\0\end{pmatrix}$ and whose 1-eigenspace is $\text{Span}\left\{\begin{pmatrix}3\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$.

i. (2 pts) Which **one** of the following is an eigenvector in the 1-eigenspace?

○ $\begin{pmatrix}0\\0\\0\end{pmatrix}$ ○ $\begin{pmatrix}3\\0\\-1\end{pmatrix}$ ○ $\begin{pmatrix}7\\0\\1\end{pmatrix}$ ● $\begin{pmatrix}6\\-1\\1\end{pmatrix}$ ○ $\begin{pmatrix}0\\-1\\1\end{pmatrix}$

ii. (2 points) Which **one** of the following is in $(\text{Nul }A)^\perp$?

● $\begin{pmatrix}4\\1\\1\end{pmatrix}$ ○ $\begin{pmatrix}1 & -4 & 0\end{pmatrix}$ ○ $\begin{pmatrix}-1\\4\\0\end{pmatrix}$ ○ $\begin{pmatrix}1\\-4\\0\end{pmatrix}$ ○ $\begin{pmatrix}3\\-1\\0\end{pmatrix}$

(d) (2 points) Suppose $T : \mathbf{R}^a \to \mathbf{R}^b$ is a linear transformation. Which **one** of the following statements guarantees that T is onto?

● For each y in $\mathbf{R}^b$, there is at least one x in $\mathbf{R}^a$ so that $T(x) = y$.

○ For each x in $\mathbf{R}^a$, there is at least one y in $\mathbf{R}^b$ so that $T(x) = y$.

○ For each x in $\mathbf{R}^a$, there is at most one y in $\mathbf{R}^b$ so that $T(x) = y$.

Problem 3 Solution.

(a) The matrix $A = \begin{pmatrix} -2 & -6 \\ 1 & 3 \end{pmatrix}$ is the only choice that satisfies $A \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) To get the final matrix, we :

- First swap R2 and R3. The new determinant is -1.
- Next, scale the new R2 by 3 and the new R3 by -4. The new determinant is $(-1)(3)(-4) = 12$.
- Finally, do a row-replacement to add 5R2 to R3 (does not change the det, it stays 12).

Therefore, our final answer is 12.

(c) (i) The vectors $v_1 = \begin{pmatrix} 3 \\ -1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ form a basis for the 1-eigenspace. Note $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is never an eigenvector, so it is incorrect. The only other choice that is a linear combination of v_1 and v_2 is $\begin{pmatrix} 6 \\ -1 \\ 1 \end{pmatrix}$, which is $2v_1 + v_2$.

(ii) For a vector x to be in $(\text{Nul A})^{\perp}$, it must be orthogonal to $\begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix}$. The only choice that satisfies $x \cdot \begin{pmatrix} 1 \\ -4 \\ 0 \end{pmatrix} = 0$ is $\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$.

(d) The bubbled in choice is nearly verbatim the definition of onto.

4. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 pts) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be a linear transformation that satisfies $T\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$.

Which **one** of the following could be the standard matrix A for T?

● $\begin{pmatrix}1&0&0&0\\0&1&1&0\\0&0&1&0\end{pmatrix}$ ○ $\begin{pmatrix}0&1&0\\0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ ○ $\begin{pmatrix}1&0&0&0\\1&1&0&0\\0&0&0&0\end{pmatrix}$ ○ $\begin{pmatrix}0&0&1&0\\1&0&0&0\\0&0&0&0\end{pmatrix}$

(b) (2 points) Let $T : \mathbf{R}^4 \to \mathbf{R}^3$ be the linear transformation given by

$$T(x, y, z, w) = (x - y,\ z - x,\ w - y).$$

Which **one** of the following statements is true?

○ T is one-to-one and onto.

○ T is one-to-one but not onto.

● T is onto but not one-to-one.

○ T is neither one-to-one nor onto.

(c) (4 points) Suppose A is a 5×5 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(2 - \lambda)^3(5 - \lambda).$$

Which of the following statements must be true? Fill in the bubble for all that apply.

● The column space of A is 4-dimensional.

● $\dim(\mathrm{Nul}(A - 5I)) = 1$.

● $\det(A) = 0$.

○ The equation $Ax = -2x$ has infinitely many solutions.

(d) (2 pts) Let $W = \mathrm{Span}\left\{\begin{pmatrix}1\\-1\\2\end{pmatrix}, \begin{pmatrix}0\\1\\3\end{pmatrix}\right\}$. Which **one** of these is a basis for $W^{\perp}$?

○ $\left\{\begin{pmatrix}0\\-3\\1\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}-1\\1\\-1\end{pmatrix}\right\}$ ● $\left\{\begin{pmatrix}-5\\-3\\1\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}-2\\0\\1\end{pmatrix}\right\}$

○ $\left\{\begin{pmatrix}1\\0\\5\end{pmatrix}\right\}$ ○ $\left\{\begin{pmatrix}-2\\0\\1\end{pmatrix}, \begin{pmatrix}0\\-3\\1\end{pmatrix}\right\}$ ○ $\{(1\ \ -1\ \ 2), (0\ \ 1\ \ 3)\}$

Problem 4 Solution.

(a) Since T has domain $\mathbf{R}^4$ and codomain $\mathbf{R}^3$, so A is 3×4.

Also, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = T(e_1 + e_2) = T(e_1) + T(e_2)$, which is the sum of the first two columns of A. The only option whose first two columns sum to $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. Alternatively, we could have just computed $A(e_1 + e_2)$ for each of the three 3×4 choices and gotten the same answer.

(b) We can compute that the matrix for T is $A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix}$. Through a bit of row-reduction we see A has 3 pivots, so it has a pivot in every row but does **not** have a row in every column. Therefore, T is onto but not one-to-one.

(c) (i) True: the null space of A is the 0-eigenspace. From the characteristic polynomial, $\lambda = 0$ has algebraic multiplicity 1 and therefore also has geometric multiplicity 1, which means the null space of A is one-dimensional. Therefore, by the Rank Theorem, the 5×5 matrix A has a 4-dimensional column space.

(ii) True: since $\lambda = 5$ has algebraic multiplicity 1, it must also have geometric multiplicity 1, which means that the null space of $A - 5I$ is 1-dimensional.

(iii) True: since $\lambda = 0$ is an eigenvalue, we know A is not invertible, so $\det(A) = 0$.

(iv) False: since $\lambda = -2$ is not an eigenvalue of A, the equation $Ax = -2x$ has only the trivial solution.

(d) It is a standard fact from chapter 6 that if $W^\perp$ is the null space of the matrix whose rows are the two basis vectors of W.

$$\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 1 & 3 \end{array}\right) \xrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 3 \end{array}\right).$$

This gives $x_1 + 5x_3 = 0$ and $x_2 + 3x_3 = 0$ with x_3 free, so: $x_1 = -5x_3$, $x_2 = -3x_3$, and x_3 is free. Therefore, a basis for $W^\perp$ is $\left\{ \begin{pmatrix} -5 \\ -3 \\ 1 \end{pmatrix} \right\}$.

5. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first reflects each vector $\begin{pmatrix} x \\ y \end{pmatrix}$ across the line $y = -x$, then performs orthogonal projection of the vector onto the x-axis. Which **one** of the following statements is true?

- ○ The eigenvalues of the standard matrix for T are 0 and 1.
- ○ The vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is in the range of T.
- ● $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) (2 points) Which **one** of the following matrices is **not** invertible?

- ○ The 2×2 matrix A that rotates vectors by $50°$.
- ● The matrix $A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ 3 & 2 & 3 \end{pmatrix}$.
- ○ Any 3×3 matrix that has characteristic polynomial $(1 - \lambda)^3$.
- ○ The standard matrix A for the linear transformation $T(x, y) = (x - 3y,\ 3x - y)$.

(c) (2 points) Let A be an 5×3 matrix with columns v_1, v_2, and v_3. Which **one** of the following statements must be true?

- ○ $\left(\text{Span}\{v_1, v_2, v_3\}\right)^\perp = \text{Nul}(A)$.
- ○ The row space of A is a subspace of $\mathbf{R}^5$.
- ● If the columns of A are linearly independent, then $\dim((\text{Col } A)^\perp) = 2$.
- ○ $(\text{Nul } A)^\perp = \text{Col } (A)$.

(d) (4 points) The matrix $A = \begin{pmatrix} 0.55 & 0.35 \\ 0.45 & 0.65 \end{pmatrix}$ is positive-stochastic and its steady-state vector is $\begin{pmatrix} 7/16 \\ 9/16 \end{pmatrix}$. Which of the following statements are true? Fill in the bubble for all that apply.

- ● $A^n \begin{pmatrix} 32 \\ 0 \end{pmatrix}$ approaches $\begin{pmatrix} 14 \\ 18 \end{pmatrix}$ as n gets very large.
- ● A is invertible.
- ● $Av = v$ for every vector v in the span of $\begin{pmatrix} 7 \\ 9 \end{pmatrix}$.
- ○ $\text{Nul}(A - I) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$.

Problem 5 Solution.

(a) The matrix for T is $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$. Note the matrix for reflection is on the **right** because it is the **first** operation that would be applied to a vector v, and the projection matrix is on the left because it is the second operation applied.
(i) is False: the only eigenvalue of A is $\lambda = 0$.
(ii) is False: the last operation is projection onto the x-axis, so every vector in the range of T must have a "0" in its second entry.
(iii) is True: just multiply A by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and you will get $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, or just apply T to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and you will see it first goes to to $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, then gets projected to $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(b) We have emphasized that rotations are invertible; any matrix with characteristic polynomial $(1-\lambda)^3$ must be invertible because it does **not** have $\lambda = 0$ as an eigenvalue; the transformation T is clearly invertible because its matrix will have a nonzero determinant; however, if we do a couple steps of row-reduction on the matrix A that is given to us, we get $\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$.

(c) (i) is False and does not even make sense, since that span's orthogonal complement is a subspace of $\mathbf{R}^5$ but the null space of A is a subspace of $\mathbf{R}^3$.
(ii) is False: the row space of A is the column space of the 3×5 matrix A^T, which is a subspace of $\mathbf{R}^3$.
(iii) is True: if the 3 columns of A are linearly independent, then we see

$$5 = \dim(\text{Col } A) + \dim((\text{Col } A)^\perp) = 3 + \dim((\text{Col } A)^\perp), \quad \text{so} \quad \dim((\text{Col } A)^\perp) = 2.$$

(iv) is False: in fact, $(\text{Nul } A)^\perp$ is a subspace of $\mathbf{R}^3$ but $\text{Col}(A)$ lives in $\mathbf{R}^5$.

(d) (i) is True: since the sum of entries in v is 32, we know $A^n v$ approaches $32 \begin{pmatrix} 7/16 \\ 9/16 \end{pmatrix} = \begin{pmatrix} 14 \\ 18 \end{pmatrix}$ as n gets large.

(ii) is True: the two columns of A are clearly linearly independent, since neither is a scalar multiple of the other.

(iii) is True: the 1-eigenspace is spanned by the steady-state vector $w = \begin{pmatrix} 7/16 \\ 9/16 \end{pmatrix}$ the Perron-Frobenius Theorem, which is the same as the span of $\begin{pmatrix} 7 \\ 9 \end{pmatrix}$.

(iv) is False: $\text{Nul}(A - I)$ is the 1-eigenspace of A, which is a line by the Perron-Frobenius Theorem. Even if A were not "positive" stochastic, we would know this is false because the steady-state vector w is a non-trivial solution to $Ax = 1x$.

6. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (c) are unrelated.

(a) (4 points) In the space provided below, write a single 3×3 matrix A satisfying **all** of the following properties:

- A is in reduced row echelon form.
- A has exactly one pivot.
- $A \begin{pmatrix} 6 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

$A =$

(b) (3 points) In the space provided below, write a single matrix A that satisfies

$(\text{Nul } A)^{\perp} = \text{Span}\left\{ \begin{pmatrix} 7 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$A =$

(c) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector in $\mathbf{R}^3$ whose orthogonal decomposition with respect to W is $x = x_W + x_{W^{\perp}}$ where

$$x_W = \begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix} \quad \text{and} \quad x_{W^{\perp}} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}.$$

(i) (1 point) What is the closest vector to x in W?

○ $\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} 6 \\ -2 \\ 1 \end{pmatrix}$ ● $\begin{pmatrix} 5 \\ -1 \\ 3 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

○ $\begin{pmatrix} -4 \\ 0 \\ -5 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$ ○ $\begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$ ○ none of these

(ii) (1 point) What is the distance from x to W?

○ 0 ○ 2 ● $\sqrt{6}$ ○ $\sqrt{7}$ ○ 3 ○ $\sqrt{35}$

○ 6 ○ $\sqrt{41}$ ○ 35 ○ 41 ○ 42

(iii) (1 point) What is the length of x?

○ $\sqrt{6}$ ○ $\sqrt{7}$ ○ 3 ○ $\sqrt{35}$ ○ 6 ○ 7

○ $\sqrt{38}$ ● $\sqrt{41}$ ○ 35 ○ 41 ○ 42

Problem 6 Solution.

(a) The requirements that A is in RREF and has one pivot mean that the first row's leading entry is 1, and that the second and third rows are all zeros.

Since $A\begin{pmatrix}6\\-1\\3\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$, the first row $(x\ y\ z)$ of A must satisfy $6x-y+3z=0$, which gives us the easy condition to check: $y=6x+3z$. Some correct answers are below.

$$A=\begin{pmatrix}1&6&0\\0&0&0\\0&0&0\end{pmatrix},\quad A=\begin{pmatrix}1&9&1\\0&0&0\\0&0&0\end{pmatrix},\quad A=\begin{pmatrix}1&3&-1\\0&0&0\\0&0&0\end{pmatrix},\quad A=\begin{pmatrix}0&1&1/3\\0&0&0\\0&0&0\end{pmatrix},\quad \text{etc.}$$

(b) Since $\text{Row}(A)=(\text{Nul }A)^{\perp}$, we just need to write a matrix whose row space is the span of $\begin{pmatrix}7\\1\\0\end{pmatrix}$. This means every row of A must be a scalar multiple of that vector, and A cannot be the zero matrix. Note A must have 3 columns, but A does not need to be a square matrix. It can have just one row if you wish! There are many possible correct answers, such as

$$A=\begin{pmatrix}7&1&0\end{pmatrix},\quad A=\begin{pmatrix}-7&-1&0\end{pmatrix},\quad A=\begin{pmatrix}14&2&0\end{pmatrix},\quad A=\begin{pmatrix}28&4&0\end{pmatrix}.$$

and

$$A=\begin{pmatrix}7&1&0\\0&0&0\\0&0&0\end{pmatrix},\quad A=\begin{pmatrix}7&1&0\\-14&-2&0\\0&0&0\end{pmatrix},\quad A=\begin{pmatrix}14&2&0\\-7&-1&0\\28&4&0\end{pmatrix},\quad A=\begin{pmatrix}14&2&0\\28&4&0\\70&10&0\end{pmatrix}.$$

(c) (i) The closest vector to x in W is x_W, which is $\begin{pmatrix}5\\-1\\3\end{pmatrix}$.

(ii) The distance from x to W is the length of $x_{W^\perp}$, which is

$$||x_{W^\perp}||=\sqrt{1^2+(-1)^2+(-2)^2}=\sqrt{6}.$$

(iii) We find $x=x_W+x_{W^\perp}=\begin{pmatrix}5+1\\-1-1\\3-2\end{pmatrix}=\begin{pmatrix}6\\-2\\1\end{pmatrix}$, so

$$||x||=\sqrt{6^2+(-2)^2+1^2}=\sqrt{41}.$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. If you need extra space for your work, please use the last page of the exam and indicate this clearly.

For this page, let $A = \begin{pmatrix} -7 & -18 & 9 \\ 0 & 2 & 0 \\ -3 & -6 & 5 \end{pmatrix}$. The only eigenvalues of A are $\lambda = 2$ and $\lambda = -4$.

(a) (6 points) Find a basis for each of the two eigenspaces of A. Enter your answers in the boxes marked below.

Basis for 2-eigenspace: $\left\{ \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right\}$

Basis for (-4)-eigenspace: $\left\{ \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix} \right\}$

Solution: $\underline{\lambda = 2}$: $A - 2I = \begin{pmatrix} -9 & -18 & 9 \\ 0 & 0 & 0 \\ -3 & -6 & 3 \end{pmatrix} \xrightarrow[\text{then } R_3 = R_3 + 3R_1]{R_1 = -R_1/9} \begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ This gives us $x_1 + 2x_2 - x_3$. Therefore, $x_1 = -2x_2 + x_3$ with x_2 and x_3 free. $(x_1, x_2, x_3) = (-2x_2 + x_3, x_2, x_3) = x_2(-2, 1, 0) + x_3(1, 0, 1)$.

$$\underline{\lambda = -4} : A + 4I = \begin{pmatrix} -3 & -18 & 9 \\ 0 & 6 & 0 \\ -3 & -6 & 9 \end{pmatrix} \xrightarrow{R_1 = -R_1/3} \begin{pmatrix} 1 & 6 & -3 \\ 0 & 6 & 0 \\ -3 & -6 & 9 \end{pmatrix} \xrightarrow[R_3 = R_3 + R_1]{R_3 = R_3 + 3R_1} \begin{pmatrix} 1 & 6 & -3 \\ 0 & 6 & 0 \\ 0 & 12 & 0 \end{pmatrix}$$

$$\xrightarrow[\text{then } R_3 = R_3 - 12R_2,\ R_1 = R_1 - 6R_2]{R_2 = R_2/6} \begin{pmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This gives us $x_1 - 3x_3 = 0$ and $x_2 = 0$ with x_3 free, so $x_1 = 3x_3$ we get $(x_1, x_2, x_3) = (3x_3, 0, x_3) = x_3(3, 0, 1)$.

(b) (3 points) A is diagonalizable. Write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.
Solution: Many answers possible. We need C to be a matrix of linearly independent eigenvectors, and D is the corresponding diagonal matrix of eigenvalues.

$$C = \begin{pmatrix} -2 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -4 \end{pmatrix}, \quad \text{or}$$

$$C = \begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{etc.}$$

(c) (1 pt) Write the eigenvalues of A^2 below. You do not need to show your work.
The eigenvalues are 4 and 16.

Solution: A^2 is diagonalizable and $A^2 = CDC^{-1}CDC^{-1} = CD^2C^{-1}$, so the eigenvalues of A^2 are the entries of D^2, which are $2^2 = 4$ and $(-4)^2 = 16$.

8. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (5 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors by $90°$ counterclockwise, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation given by

$$U(x, y) = (5x + y, -8y).$$

Find the standard matrix A for the transformation $U \circ T$. Evaluate all trigonometric functions you write. Do not leave your answer in terms of sine and cosine.

Solution: $(U \circ T)(x) = U(T(x))$, so we **first** apply T and **then** apply U.

This means that the matrix for T goes on the **right** and the matrix for U goes on the **left**.

$$\begin{pmatrix} 5 & 1 \\ 0 & -8 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -5 \\ -8 & 0 \end{pmatrix}.$$

Alternatively, our answer is the matrix whose first column is $(U \circ T)(e_1)$ and second column is $(U \circ T)(e_2)$.

- To get $(U \circ T)(e_1)$, we rotate $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to get $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then perform U to get $\begin{pmatrix} 1 \\ -8 \end{pmatrix}$.
- To get $(U \circ T)(e_2)$, we rotate $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ to get $\begin{pmatrix} -1 \\ 0 \end{pmatrix}$, then perform U to get $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$.

(b) (5 points) Find a matrix A that satisfies

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = -5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad A \begin{pmatrix} 4 \\ 3 \end{pmatrix} = 4 \begin{pmatrix} 4 \\ 3 \end{pmatrix}.$$

Enter your answer in the space to the right.

Solution: The (-5)-eig. is the span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and the 4-eig. is the span of $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$. By the Diagonalization Theorem, $A = CDC^{-1}$, where C is formed by eigenvectors and D is the diagonal matrix formed by the corresponding eigenvalues. With $C = \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix}$ and $D = \begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix}$, we get $C^{-1} = \frac{1}{-1} \begin{pmatrix} 3 & -4 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ 1 & -1 \end{pmatrix}$ and

$$\begin{aligned} A = CDC^{-1} &= \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix} \left[\begin{pmatrix} -5 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -3 & 4 \\ 1 & -1 \end{pmatrix} \right] \\ &= \begin{pmatrix} 1 & 4 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 15 & -20 \\ 4 & -4 \end{pmatrix} = \begin{pmatrix} 31 & -36 \\ 27 & -32 \end{pmatrix}. \end{aligned}$$

9. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (4 points) Find all values of c so that the following vectors v_1, v_2, and v_3 are linearly **independent**:

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 2 \\ 6 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 3 \\ c \\ -8 \end{pmatrix}.$$

Solution: We row-reduce to find when the corresponding matrix has a pivot in every column.

$$\begin{pmatrix} 1 & 2 & 3 \\ -1 & 2 & c \\ -1 & 6 & -8 \end{pmatrix} \xrightarrow[R_3=R_3+R_1]{R_2=R_2+R_1} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & c+3 \\ 0 & 8 & -5 \end{pmatrix} \xrightarrow{R_3=R_3-2R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & c+3 \\ 0 & 0 & -11-2c \end{pmatrix}.$$

The matrix has a pivot in every column if and only if $-11-2c \neq 0$, so $-2c \neq 11$. In other words,

$$c \neq \frac{-11}{2}.$$

(b) (6 points) Let $W = \text{Span}\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}$, and let $x = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$. Find the orthogonal decomposition of x with respect to W.
In other words, find x_W in W and $x_{W^\perp}$ in $W^\perp$ satisfying $x = x_W + x_{W^\perp}$. Write your answers in the space below and simplify them completely.

$x_W = \begin{pmatrix} -4/17 \\ -1/17 \end{pmatrix}$ and $x_{W^\perp} = \begin{pmatrix} -13/17 \\ 52/17 \end{pmatrix}$.

Solution: With $u = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, we can compute the matrix B for orthogonal projection onto $W = \text{Span}\{u\}$.

$$B = \frac{1}{u \cdot u} u u^T = \frac{1}{4^2+1^2} \begin{pmatrix} 4 \\ 1 \end{pmatrix} \begin{pmatrix} 4 & 1 \end{pmatrix} = \frac{1}{17} \begin{pmatrix} 16 & 4 \\ 4 & 1 \end{pmatrix}.$$

Therefore,

$$x_W = Bx = \frac{1}{17} \begin{pmatrix} 16 & 4 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} -4/17 \\ -1/17 \end{pmatrix}$$

and

$$x_{W^\perp} = x - x_W = \begin{pmatrix} -1 \\ 3 \end{pmatrix} - \begin{pmatrix} -4/17 \\ -1/17 \end{pmatrix} = \begin{pmatrix} -13/17 \\ 52/17 \end{pmatrix}.$$

10. Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

In this problem, we use the usual convention of (x, y) to denote points in $\mathbf{R}^2$.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(-1, 6), \qquad (1, -7), \qquad (3, -2).$$

Enter your answer below:

$$y = -2x + 1.$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

No line goes through all three points. The corresponding (inconsistent) system is

$$6 = M(-1) + B$$

$$-7 = M(1) + B$$

$$-2 = M(3) + B.$$

The corresponding matrix equation is $Ax = b$ where $A = \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 6 \\ -7 \\ -2 \end{pmatrix}$.

We solve $A^T A\hat{x} = A^T b$.

$$A^T A = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 11 & 3 \\ 3 & 3 \end{pmatrix} \qquad A^T b = \begin{pmatrix} -1 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 6 \\ -7 \\ -2 \end{pmatrix} = \begin{pmatrix} -19 \\ -3 \end{pmatrix}$$

$$\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 11 & 3 & -19 \\ 3 & 3 & -3 \end{array} \right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 11 & 3 & -19 \end{array} \right) \xrightarrow{R_2 = R_2 - 11R_1} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & -8 & -8 \end{array} \right)$$

$$\xrightarrow{R_2 = -R_2/8} \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 1 & 1 \end{array} \right) \xrightarrow{R_1 = R_1 - R_2} \left(\begin{array}{cc|c} 1 & 0 & -2 \\ 0 & 1 & 1 \end{array} \right).$$

Thus $\hat{x} = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$. The line is

$$y = -2x + 1.$$

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If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.