Math 1553 Exam 3, SOLUTIONS, Fall 2025, Version B

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25 AM) Kim (B, 8:00 AM) Kim (C, 9:00 AM)

Callis (D, 10:00 AM) Short (E, 9:30 AM) Shi (F, 11:00 AM)

Short (H, 12:30 PM) He (I, 2:00 PM) Stokolosa (L, 3:30 PM)

Van Why (M, 3:30 PM) Yap (N, 5:00 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- Calculators and cell phones are not allowed. Aids of any kind (notes, text, etc.) are not allowed. If you use pen, you must use black ink. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, November 12.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) Suppose A is an $n \times n$ matrix that is not invertible. Then the bottom row of the RREF of A must be a row of zeros.

   ● True

   ○ False

   (b) If A is an $n \times n$ matrix and $\det(A) = 0$, then $\lambda = 0$ must be an eigenvalue of A.

   ● True

   ○ False

   (c) Suppose A is a diagonalizable $n \times n$ matrix and b is a vector in $\mathbf{R}^n$. Then b must be in one of the eigenspaces of A.

   ○ True

   ● False

   (d) Suppose A is an $n \times n$ matrix and that u and v are eigenvectors of A corresponding to different eigenvalues of A. Then $u + v$ cannot be an eigenvector of A.

   ● True

   ○ False

   (e) If A is a diagonalizable $n \times n$ matrix, then A^3 must also be diagonalizable.

   ● True

   ○ False

Problem 1 Solution.

(a) True. If A is not invertible, then its RREF will have fewer than n pivots, so there is no pivot in the final row of the RREF, therefore it is just a row of zeros.

(b) True: if $\det(A) = 0$ then A is not invertible, which means $Ax = 0x$ has infinitely many solutions, so $\lambda = 0$ is an eigenvalue of A. There are other ways to see this. For example, since $\det(A) = 0$ we know that $\det(A - 0I) = 0$ which means that $A - 0I$ is not invertible, so $\lambda = 0$ is an eigenvalue of A.

(c) False: if $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ then A is diagonalizable (in fact, diagonal!). However,

$$A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

which is not a scalar multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, so the nonzero vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector of A, therefore it is not in any eigenspace of A.

(d) True: if u and v are eigenvectors in different eigenspaces, then their sum will not be an eigenvector. For example, if $Au = 3u$ and $Av = 7v$, then $u + v$ is not an eigenvector because

$$A(u + v) = Au + Av = 3u + 7v$$

which is not a scalar multiple of $u + v$.

(e) True: if $A = CDC^{-1}$ for an invertible $n \times n$ C and a diagonal D, then $A^3 = CD^3C^{-1}$ for the same C and the diagonal matrix D^3.

2. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 points) Find A^{-1} for $A = \begin{pmatrix} 1 & -4 \\ 3 & -13 \end{pmatrix}$.

$\bigcirc$ $A^{-1} = \begin{pmatrix} 13 & -3 \\ 4 & -1 \end{pmatrix}$ $\bigcirc$ $A^{-1} = \begin{pmatrix} -13 & 4 \\ -3 & 1 \end{pmatrix}$ $\bullet$ $A^{-1} = \begin{pmatrix} 13 & -4 \\ 3 & -1 \end{pmatrix}$

$\bigcirc$ $A^{-1} = \frac{1}{25}\begin{pmatrix} 13 & -4 \\ 3 & -1 \end{pmatrix}$ $\bigcirc$ $A^{-1} = \frac{1}{25}\begin{pmatrix} 13 & -3 \\ 4 & -1 \end{pmatrix}$ $\bigcirc$ $A^{-1} = \frac{1}{25}\begin{pmatrix} -13 & 4 \\ -3 & 1 \end{pmatrix}$

$\bigcirc$ none of these

(b) (4 points) Suppose A and B are **invertible** $n \times n$ matrices. Which of the following statements must be true? Fill in the bubble for all that apply.

$\bigcirc$ If M is the matrix obtained by multiplying the first row of A by 3, then $\det(A) = 3\det(M)$.

$\bullet$ The matrix AB must be invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

$\bullet$ If n is even, then $\det(A) = \det(-A)$.

$\bigcirc$ Performing a row replacement on A does not change the eigenvalues of A.

(c) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} a & b & c \\ 2g-3d & 2h-3e & 2i-3f \\ 4g & 4h & 4i \end{pmatrix}$.

$\bigcirc$ 2 $\bigcirc$ 3 $\bigcirc$ 4 $\bigcirc$ 6 $\bigcirc$ 8 $\bigcirc$ 12

$\bigcirc$ -2 $\bigcirc$ -3 $\bigcirc$ -4 $\bigcirc$ -6 $\bigcirc$ -8 $\bullet$ -12

$\bigcirc$ 24 $\bigcirc$ -24 $\bigcirc$ none of these $\bigcirc$ not enough info

(d) (2 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 1 & 0 & 8 \\ 0 & 1 & 0 \\ -1 & 0 & 7 \end{pmatrix}$.

Fill in the bubble for your answer below.

$\bigcirc$ 1 and 8 $\bigcirc$ 1 and 7 $\bigcirc$ 1, -1, and 8 $\bullet$ 1, 3, and 5

$\bigcirc$ 1, -1, and 7 $\bigcirc$ 1, 2, and 6 $\bigcirc$ 1, -2, and -6

$\bigcirc$ 1, -3, and -5 $\bigcirc$ none of these

Problem 2 Solution.

(a) For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ we know $A^{-1} = \dfrac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so here

$$A^{-1} = \frac{1}{-13-(-12)} \begin{pmatrix} -13 & 4 \\ -3 & 1 \end{pmatrix} = -\begin{pmatrix} -13 & 4 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} 13 & -4 \\ 3 & -1 \end{pmatrix}.$$

(b) (i) is false: scaling a row by a factor of 3 will scale the determinant by 3, so in fact $\det(M) = 3\det(A)$, not the other way around.

(ii) is true, a fundamental fact about inverses:

$$(AB)(B^{-1}A^{-1}) = ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$$

and similarly one could compute $(B^{-1}A^{-1})(AB) = I$.

(iii) is true: $\det(-A) = (-1)^n \det(A)$, so if n is even $(-1)^n = 1$ and $\det(-A) = \det(A)$.

(iv) the eigenvalues of the new matrix will often be different than the original. For example, the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ has eigenvalues 0 and 1, but performing the row-replacement $R_2 = R_2 + R_2$ gives us the $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ which has eigenvalues 0 and 2. Due to the poor technical phrasing of this statement, we gave everyone credit.

(c) To get the final matrix, we start with the original matrix and then:

i. Scale the third row by 4. The new determinant is 4.
ii. Scale the second row by -3 to make it $(-3d \ -3e \ -3f)$. The new determinant is $4(-3) = -12$.
iii. Do a row replacement by adding $2R_3$ to R_2, making the second row $(2g-3d \ \ 2h-3e \ \ 2i-3f)$. The determinant does not change.

Therefore, the final determinant is -12.

(d) The cofactor expansion along the second row of $\det \begin{pmatrix} 1-\lambda & 0 & 8 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 7-\lambda \end{pmatrix}$ gives

$$\begin{aligned} 0 &= 0 + (1-\lambda)(-1)^{2+2} \det \begin{pmatrix} 1-\lambda & 8 \\ -1 & 7-\lambda \end{pmatrix} + 0 \\ &= (1-\lambda)\Big[(1-\lambda)(7-\lambda)+8\Big] = (1-\lambda)\Big[\lambda^2 - 8\lambda + 15\Big] = (1-\lambda)(\lambda-3)(\lambda-5), \end{aligned}$$

so the eigenvalues are $\lambda = 1$, $\lambda = 3$, and $\lambda = 5$.

3. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

   (a) (2 points) Find all real values of c (if there are any) so that $\det \begin{pmatrix} 1 & c & 2 \\ 0 & -1 & 1 \\ 2 & 1 & c \end{pmatrix} = 0.$

   Fill in the bubble for your answer below.

   ○ $c = 0$ only ○ $c = -1$ only ○ $c = 3$ only ● $c = -3$ only

   ○ All c except -1 ○ All c except 3 ○ All c except -3

   ○ There is no value of c so that the determinant is 0 ○ none of these

   (b) (3 points) Let A be the 2×2 matrix that reflects each $\begin{pmatrix} x \\ y \end{pmatrix}$ across the line $y = -5x$. Which of the following statements are true? Fill in the bubble for all that apply.

   ● The 1-eigenspace of A is the line $y = -5x$.

   ● A is diagonalizable.

   ● $Av = -v$ for all v in the span of $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

   (c) (2 points) Which **one** of the following matrices has the property that its 3-eigenspace is a plane?

   ● $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

   ○ $\begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$ ○ none of these

   (d) (3 points) Suppose A is an $n \times n$ matrix and λ is a real number. Which of the following conditions guarantee that λ is an eigenvalue of A? Fill in the bubble for all that apply.

   ○ The set $\{Ax, \lambda x\}$ is linearly dependent for some vector x.

   ● The null space of $A - \lambda I$ is 1-dimensional.

   ● The equation $Ax = \lambda x$ has infinitely many solutions.

Problem 3 Solution.

(a) We compute

$$0 = 1(-c-1) - c(0-2) + 2(0+2), \qquad 0 = -c-1+2c+4, \qquad 0 = c+3, \qquad c = -3.$$

(b) This problem uses the standard facts about 2×2 reflection matrices that that we have emphasized:

- The eigenvalues are 1 and -1.
- The 1-eigenspace is the line of reflection.
- The (-1)-eigenspace is the line through the origin that is perpendicular to the line of reflection.

(i) is true: the 1-eigenspace is the line of reflection, namely $y = -5x$.

(ii) is true: A is a 2×2 real matrix with 2 different real eigenvalues, so it is automatically diagonalizable. We did not need to know the entries of the matrix in order to do this.

(iii) is true: the (-1)-eigenspace is the line $y = x/5$, which is the span of $\begin{pmatrix} 5 \\ 1 \end{pmatrix}$.

(c) Each 2×2 matrix has 3-eigenspace that is a line, and the 3×3 matrix $3I$ has all of $\mathbf{R}^3$ as its 3-eigenspace.

The matrices $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}$ and $\begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}$ have a 3-eigenspace that is only a line, because $A - 3I$ has exactly two pivots (so $(A-3I)x = 0$ has only one free variable).

The only remaining choice is the correct answer: $\begin{pmatrix} 3 & 1 & 2 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Note that $A - 3I$ is the matrix $\begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ which has exactly one pivot, so $(A - 3I)x = 0$ has exactly two free variables, therefore the 3-eigenspace is a plane.

(d) (i) does not guarantee λ is an eigenvalue, and in fact $\{A(0), \lambda(0)\}$ is linearly dependent for $x = 0$ no matter what A and λ are.

(ii) guarantees λ is an eigenvalue because it means $(A - \lambda I)x = 0$ has a non-trivial solution, therefore $Ax = \lambda x$ has a non-trivial solution.

(iii) guarantees λ is an eigenvalue, it is almost the definition of eigenvalue.

4. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 points) Suppose A and B are 2×2 matrices with $\det(A) = 3$ and $\det(B) = 5$. Find $\det(2A^{-1}B)$. Fill in the bubble for your answer below.

○ 5 ○ $\frac{10}{3}$ ● $\frac{20}{3}$ ○ 30 ○ 6 ○ none of these

(b) (4 points) Let $A = \begin{pmatrix} -4 & 3 \\ -1 & -7 \end{pmatrix} \begin{pmatrix} 1/4 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -4 & 3 \\ -1 & -7 \end{pmatrix}^{-1}$.

(i) Find $A \begin{pmatrix} -4 \\ -1 \end{pmatrix}$. Fill in the bubble for your answer below.

○ $\begin{pmatrix} 1 \\ 8 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ ● $\begin{pmatrix} -1 \\ -1/4 \end{pmatrix}$ ○ $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 1/4 \end{pmatrix}$

○ $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ ○ $\begin{pmatrix} 3/4 \\ -7/4 \end{pmatrix}$ ○ none of these

(ii) Find $A^5 \begin{pmatrix} 3 \\ -7 \end{pmatrix}$. Fill in the bubble for your answer below.

○ $\begin{pmatrix} -8 \\ -2 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} -2 \\ -1/2 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ 1/2 \end{pmatrix}$

○ $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ ● $\begin{pmatrix} -3 \\ 7 \end{pmatrix}$ ○ $\begin{pmatrix} -3/4 \\ 7/4 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ ○ $\begin{pmatrix} 15 \\ -35 \end{pmatrix}$

(c) (2 points) Suppose A is a 5×5 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(2 - \lambda)^3(4 - \lambda).$$

Which **one** of the following statements is true?

○ It is possible for the null space of A to be a plane.

○ $\dim(\mathrm{Nul}(A - 2I))$ must equal 3.

○ $\det(A) = 8$.

● If the 2-eigenspace of A has dimension 3, then A must be diagonalizable.

(d) (2 points) Which one of the following matrices is **invertible** but **not diagonalizable**?

○ $\begin{pmatrix} 5 & -1 \\ 0 & 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$ ● $\begin{pmatrix} 5 & 6 \\ 0 & 5 \end{pmatrix}$ ○ $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$ ○ $\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix}$

Problem 4 Solution.

(a) For 2×2 matrices, multiplying the matrix by 2 will multiply each of the two rows by 2, so the determinant is multiplied by $2^2 = 4$.

$$\det(2A^{-1}B) = 2^2 \det(A^{-1}B) = 4 \cdot \frac{1}{\det(A)} \cdot \det(B) = 4(1/3)(5) = \frac{20}{3}.$$

(b) By the Diagonalization Theorem, we see $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ is an eigenvector for the eigenvalue $\lambda = 1/4$ and that $\begin{pmatrix} 3 \\ -7 \end{pmatrix}$ is an eigenvector for the eigenvalue -1.

(i) $A \begin{pmatrix} -4 \\ -1 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} -4 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1/4 \end{pmatrix}$.

(ii) $A^5 \begin{pmatrix} 3 \\ -7 \end{pmatrix} = (-1)^5 \begin{pmatrix} 3 \\ -7 \end{pmatrix} = -\begin{pmatrix} 3 \\ -7 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$.

(c) (i) is not true. The algebraic multiplicity of $\lambda = 0$ is 1 since $(0 - \lambda)$ has exponent 1 in the characteristic polynomial, so the null space of A is automatically a line.

(ii) is not true. From the fact that $\lambda = 2$ has algebraic multiplicity 3, all that we know is that $1 \leq \dim(\mathrm{Nul}(A - 2I)) \leq 3$.

(iii) is not true: A is not invertible since $\lambda = 0$ is an eigenvalue, so $\det(A) = 0$.

(iv) is true: If the 2-eigenspace is 3-dimensional, then since we already know that the 0-eigenspace and 4-eigenspace are 1-dimensional (0 and 4 have alg. mult. 1, thus geo. mult. 1), we get 5 linearly independent eigenvectors in $\mathbf{R}^5$, therefore A is diagonalizable.

(d) From determinants, the only two invertible matrices are $\begin{pmatrix} 5 & -1 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 5 & 6 \\ 0 & 5 \end{pmatrix}$. Among these two candidates, we see $\begin{pmatrix} 5 & -1 \\ 0 & 1 \end{pmatrix}$ has two different real eigenvalues so it is a diagonalizable 2×2 matrix.

The only remaining option is $\begin{pmatrix} 5 & 6 \\ 0 & 5 \end{pmatrix}$, which is not diagonalizable since its only eigenvalue is $\lambda = 5$ but its 5-eigenspace is a line: $A - 5I = \begin{pmatrix} 0 & 6 \\ 0 & 0 \end{pmatrix}$ so the augmented system $(A - 5I \mid 0)$ will have only one free variable.

5. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. If you need extra space for your work, please use the last page of the exam and indicate this clearly.

For this page, let $A = \begin{pmatrix} 7 & -8 & 12 \\ 4 & -5 & 12 \\ 0 & 0 & 3 \end{pmatrix}$. The only eigenvalues of A are $\lambda = -1$ and $\lambda = 3$.

(a) (6 points) Find a basis for each of the two eigenspaces of A. Enter your answers in the boxes marked below.

Basis for (-1)-eigenspace: $\left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$

Basis for 3-eigenspace: $\left\{ \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} \right\}$

Solution:

$$\underline{\lambda = -1}: A + I = \begin{pmatrix} 8 & -8 & 12 \\ 4 & -4 & 12 \\ 0 & 0 & 4 \end{pmatrix} \xrightarrow[R_3 = R_3/4]{R_1 = R_1/8} \begin{pmatrix} 1 & -1 & 3/2 \\ 4 & -4 & 12 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{R_2 = R_2 - 4R_1} \begin{pmatrix} 1 & -1 & 3/2 \\ 0 & 0 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\xrightarrow[\text{finish column 3}]{R_2 = R_2/6} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ so } (A + I \mid 0) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This gives us $x_1 - x_2 = 0$ and $x_3 = 0$. Therefore, $x_1 = x_2$ with x_2 free and $x_3 = 0$, so $(x_1, x_2, x_3) = (x_2, x_2, 0) = x_2(1, 1, 0)$.

$$\underline{\lambda = 3}: A - 3I = \begin{pmatrix} 4 & -8 & 12 \\ 4 & -8 & 12 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1/4} \begin{pmatrix} 1 & -2 & 3 \\ 4 & -8 & 12 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 = R_2 - 4R_1} \begin{pmatrix} 1 & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{so } (A - 3I \mid 0) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

This gives us $x_1 - 2x_2 + 3x_3 = 0$ with x_2 and x_3 free, so $x_1 = 2x_2 - 3x_3$ and we get $(x_1, x_2, x_3) = (2x_2 - 3x_3, x_2, x_3) = x_2(2, 1, 0) + x_3(-3, 0, 1)$.

(b) (3 points) A is diagonalizable. Write an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.
Solution: Many answers possible. We need C to be a matrix of linearly independent eigenvectors, and D is the corresponding diagonal matrix of eigenvalues.

$$C = \begin{pmatrix} 1 & 2 & -3 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix}, \quad \text{or}$$

$$C = \begin{pmatrix} 2 & -3 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \text{etc.}$$

(c) (1 pt) How many solutions are there to the matrix equation $Ax = -5x$? You do not need to show your work on this part, and there is no partial credit.

○ No solutions ● Exactly one solution ○ Infinitely many solutions

Solution: We were told that the only eigenvalues of A are $\lambda = -1$ and $\lambda = 3$, so $\lambda = -5$ is NOT an eigenvalue. Therefore, the equation $Ax = -5x$ has only the trivial solution.

6. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit. Parts (a) and (b) are unrelated.

   (a) (5 points) Find the complex eigenvalues of $A = \begin{pmatrix} -2 & 1 \\ -5 & -4 \end{pmatrix}$. For the eigenvalue with **negative** imaginary part, find a corresponding eigenvector v.
   Simplify your eigenvalues as much as possible!

   The eigenvalues are: $-3+2i$ and $-3-2i$ $\qquad v = \begin{pmatrix} -1 \\ 1+2i \end{pmatrix}$.

   Solution: We can solve $\det(A - \lambda I) = 0$ directly, or we can use the shortcut:

   $$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - (-2-4)\lambda + (8+5) = \lambda^2 + 6\lambda + 13.$$

   $$\lambda = \frac{-6 \pm \sqrt{6^2 - 4(1)(13)}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = \frac{-6 \pm 4i}{2} = -3 \pm 2i.$$

   We use the 2×2 eigenvector trick to get an eigenvalue for $\lambda = -3 - 2i$.

   $$(A - \lambda I \mid 0) = \left(\begin{array}{cc|c} -2-(-3-2i) & 1 & 0 \\ -5 & -4-(-3-2i) & 0 \end{array} \right)$$
   $$= \left(\begin{array}{cc|c} 1+2i & 1 & 0 \\ -5 & -1+2i & 0 \end{array} \right) = \left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array} \right)$$

   An eigenvector is $v = \begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -1 \\ 1+2i \end{pmatrix}$, or alternatively $v = \begin{pmatrix} b \\ -a \end{pmatrix} = \begin{pmatrix} 1 \\ -1-2i \end{pmatrix}$

   Other answers are possible since any nonzero complex multiple of the above is also correct:

   $$v = \begin{pmatrix} 1-2i \\ -5 \end{pmatrix}, \qquad v = \begin{pmatrix} -1+2i \\ 5 \end{pmatrix}, \qquad \text{etc..}$$

   (b) (5 points) Find A^{-1} for $A = \begin{pmatrix} 1 & 0 & 4 \\ -2 & -1 & 0 \\ 3 & 1 & 3 \end{pmatrix}$.

   Solution:

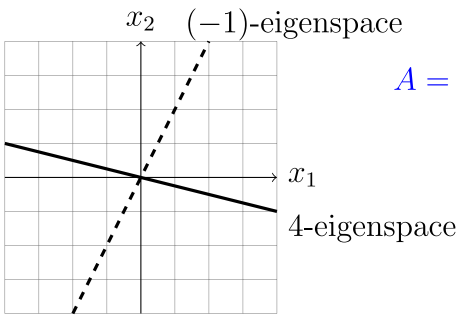
   $$\left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ -2 & -1 & 0 & 0 & 1 & 0 \\ 3 & 1 & 3 & 0 & 0 & 1 \end{array} \right) \xrightarrow[R_3 = R_3 - 3R_1]{R_2 = R_2 + 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & -1 & 8 & 2 & 1 & 0 \\ 0 & 1 & -9 & -3 & 0 & 1 \end{array} \right)$$

   $$\xrightarrow[\text{then } R_2 = -R_2]{R_3 = R_3 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -8 & -2 & -1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 \end{array} \right) \xrightarrow{R_3 = -R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 4 & 1 & 0 & 0 \\ 0 & 1 & -8 & -2 & -1 & 0 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array} \right)$$

   $$\xrightarrow[R_1 = R_1 - 4R_3]{R_2 = R_2 + 8R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 4 & 4 \\ 0 & 1 & 0 & 6 & -9 & -8 \\ 0 & 0 & 1 & 1 & -1 & -1 \end{array} \right), \quad \text{so} \quad A^{-1} = \begin{pmatrix} -3 & 4 & 4 \\ 6 & -9 & -8 \\ 1 & -1 & -1 \end{pmatrix}.$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (5 points) Find the matrix A whose 4-eigenspace is the **solid** line and whose (-1)-eigenspace is the **dashed** line graphed below. Enter your answer in the space provided. (Note: each square in the grid has sides of length 1)



$$A = \frac{1}{9}\begin{pmatrix} 31 & -20 \\ -10 & -4 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} 31/9 & -20/9 \\ -10/9 & -4/9 \end{pmatrix}.$$

Solution: The (-1)-eig. is the span of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and the 4-eig. is the span of $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$. By the Diagonalization Theorem, $A = CDC^{-1}$, where C is formed by eigenvectors and D is the diagonal matrix formed by the corresponding eigenvalues. With $C = \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$ and $D = \begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}$, we get $C^{-1} = \frac{1}{-9}\begin{pmatrix} -1 & -4 \\ -2 & 1 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}$ and

$$\begin{aligned} A = CDC^{-1} &= \begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix} \cdot \frac{1}{9}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix} \\ &= \frac{1}{9}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\left[\begin{pmatrix} -1 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\right] \\ &= \frac{1}{9}\begin{pmatrix} 1 & 4 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} -1 & -4 \\ 8 & -4 \end{pmatrix} = \frac{1}{9}\begin{pmatrix} 31 & -20 \\ -10 & -4 \end{pmatrix}. \end{aligned}$$

(b) (5 points) Find all values of c so that the matrix A below has exactly one real eigenvalue of algebraic multiplicity 2.

$$A = \begin{pmatrix} 2 & c \\ -c & 10 \end{pmatrix}.$$

Solution: We need the characteristic polynomial to be a perfect square, so that it has exactly one real root of algebraic multiplicity 2.

$$\det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 12\lambda + (20 + c^2).$$

For this to be a perfect square, we need it to be

$$(\lambda - 6)^2 = \lambda^2 - 12\lambda + 36.$$

Therefore, $20 + c^2 = 36$, so $c^2 = 16$ and therefore $c = \pm 4$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

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