Math 1553 Exam 1, SOLUTIONS, Fall 2025, Version B

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25 AM) Kim (B, 8:00 AM) Kim (C, 9:00 AM)

Callis (D, 10:00 AM) Short (E, 9:30 AM) Shi (F, 11:00 AM)

Short (H, 12:30 PM) He (I, 2:00 PM) Stokolosa (L, 3:30 PM)

Van Why (M, 3:30 PM) Yap (N, 5:00 PM)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- Calculators and cell phones are not allowed. Aids of any kind (notes, text, etc.) are not allowed. If you use pen, you must use black ink.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, September 17.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) If an augmented matrix in RREF has a pivot in every row, then its corresponding system of linear equations must be consistent.

   ○ True

   ● False

   (b) Let $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$, and $v_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$.

   Then every vector in $\mathbf{R}^3$ can be written as a linear combination of v_1, v_2, and v_3.

   ○ True

   ● False

   (c) Suppose a linear system of 3 equations in 2 variables is consistent. Then the system must have exactly one solution.

   ○ True

   ● False

   (d) If the zero vector is a solution to a matrix equation, then the matrix equation must be homogeneous.

   ● True

   ○ False

   (e) Suppose v_1 and v_2 are vectors in $\mathbf{R}^2$ and that $\text{Span}\{v_1, v_2\}$ is a line. Then there is some vector b in $\mathbf{R}^2$ so that the vector equation $x_1v_1 + x_2v_2 = b$ is inconsistent.

   ● True

   ○ False

Problem 1 Solution.

(a) False, for example $\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 1 \end{array}\right)$ has a pivot in every row, but it represents an inconsistent system.

(b) False: we row reduce the matrix with columns v_1, v_2 and v_3 and find that it has a row without a pivot, so the span of the three vectors is just a plane in $\mathbf{R}^3$.

$$\begin{pmatrix} 1 & 2 & 0 \\ -1 & -3 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_2 = R_2 + R_1} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \xrightarrow{R_3 = R_3 - R_2} \begin{pmatrix} 1 & 2 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) False, for example the system for the augmented matrix below represents 3 linear equations in 2 variables, but it has infinitely many solutions.

$$\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 2 & -2 & 0 \\ 3 & -3 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

(d) True: if $x = 0$ is a solution to a matrix equation $Ax = b$, then $b = A(0) = 0$, so the matrix equation is really just the homogeneous equation $Ax = 0$.

(e) True: since the span of v_1 and v_2 is only a line and not all of $\mathbf{R}^2$, we can certainly find a vector b that is not in $\text{Span}\{v_1, v_2\}$, so the vector equation $x_1v_1 + x_2v_2 = b$ will be inconsistent for that b. In fact, a stronger statement is true: there are infinitely many vectors that are not in the span of v_1 and v_2.

2. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (d) are unrelated.

(a) (3 points) Which of the following equations are **linear** equations in x, y, and z? Clearly fill in the bubble for all that apply.

● $10x - y - 3z = 5$

○ $2x - yz - 9z = 0$

● $3x - 5^{1/3}y + 4z = 1$

(b) (3 points) Which of the following statements are true? Clearly fill in the bubble for all that apply.

● The matrix $\left(\begin{array}{cccc|c} 0 & 1 & -3 & 0 & -1 \\ 0 & 0 & 0 & 1 & 15 \end{array}\right)$ is in RREF.

● The linear system corresponding to the augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & -3 \\ 0 & 4 & -1 \\ 0 & 8 & -2 \end{array}\right)$ has exactly one solution.

○ There is a 3×5 augmented matrix in RREF whose **bottom** row is $\left(\begin{array}{cccc|c} 0 & 1 & 0 & 0 & 0 \end{array}\right)$.

(c) (2 points) Consider the linear system for the augmented matrix $\left(\begin{array}{ccc|c} 1 & -5 & 0 & 3 \\ -2 & 10 & 1 & 4 \end{array}\right)$.
Which **one** of the following is the parametric form for the system's solution set?

○ $x_1 = 3 + 5x_2, \quad x_2 = x_2 \ (x_2 \text{ real}), \quad x_3 = 4.$

○ $x_1 = 3, \quad x_2 = 0, \quad x_3 = 4.$

● $x_1 = 3 + 5x_2, \quad x_2 = x_2 \ (x_2 \text{ real}), \quad x_3 = 10.$

○ $x_1 = 3, \quad x_2 = 0, \quad x_3 = 10.$

(d) (2 points) Consider the following linear system of equations, where h is some real number:

$$2x + 5y = -7$$
$$6x + hy = 1.$$

Which **one** of the following statements is true?

● If the system is consistent, it must have exactly one solution.

○ There is exactly one value of h that makes the system consistent.

○ If $h = 10$, then the system has infinitely many solutions.

○ The system must be consistent, regardless of the value of h.

Problem 2 Solution.

(a) The first and third equations are linear. The second equation is not linear because of the yz term. In the third equation, the $5^{1/3}$ term is just the coefficient of y, it is simply a real number times y.

(b) Statement (i) is true, since the matrix meets every condition of RREF.

Statement (ii) is true: one step of row-reduction gives $\left(\begin{array}{cc|c} \boxed{1} & 2 & -3 \\ 0 & \boxed{4} & -1 \\ 0 & 0 & 0 \end{array}\right)$, so the system has exactly one solution.

Statement (iii) is false: it is never possible for the **third** row of a matrix in RREF to have a pivot in its **second** column. The pivots in RREF must go from left to right, so if the third row has a pivot then it must be in the third column or even further to the right.

(c) Doing the row-operation $R_2 = R_2 + 2R_1$ gives us the RREF of

$$\left(\begin{array}{ccc|c} 1 & -5 & 0 & 3 \\ 0 & 0 & 1 & 10 \end{array}\right),$$

so $x_1 - 5x_2 = 3$ and $x_3 = 10$ while x_3 is free. Therefore,

$$x_1 = 3 + 5x_2, \quad x_2 = x_2 \quad (x_2 \text{ real}), \quad x_3 = 10.$$

(d) Statement (i) is true: if we row-reduce, we will find that the system is consistent if and only if $h \neq 15$, in which case the augmented matrix will have a pivot in each column except the rightmost column, therefore there will be exactly one solution.

Statement (ii) is false: the system is consistent as long as $h \neq 15$.

Statement (iii) is false: if $h = 10$, then row-reduction gives a pivot in each column except the rightmost column (since the two lines are not parallel), so the system has exactly one solution.

Statement (iv) is false: as established above, the system is inconsistent when $h = 15$.

3. On this page, you do not need to show work, and only your answers are graded. Parts (a) through (e) are unrelated.

(a) (2 points) Solve for a and b in the equation $3\begin{pmatrix} a \\ -5 \end{pmatrix} + 2\begin{pmatrix} 1 \\ b \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$.

○ $a = 2$ and $b = -3$ ○ $a = 6$ and $b = -3$ ○ $a = 7$ and $b = 8$

● $a = 2$ and $b = 8$ ○ $a = 7$ and $b = 6$ ○ none of these

(b) (2 points) Solve $Ax = b$, where $A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$.

Fill in the bubble for your answer below.

● $x = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$ ○ $x = \begin{pmatrix} -3 \\ -2 \end{pmatrix}$ ○ $x = \begin{pmatrix} -11 \\ 16 \end{pmatrix}$ ○ $x = \begin{pmatrix} -7 \\ 2 \end{pmatrix}$

(c) (2 points) Consider the vector equation $x_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ -2 \end{pmatrix} + x_3\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Which of the following describes the solution set to the vector equation?

○ a point in $\mathbf{R}^2$ ○ a line in $\mathbf{R}^2$ ○ all of $\mathbf{R}^2$

○ a point in $\mathbf{R}^3$ ○ a line in $\mathbf{R}^3$ ● a plane in $\mathbf{R}^3$

(d) (2 points) Suppose v_1 and v_2 are vectors in $\mathbf{R}^n$. Which **one** of the following statements **must** be true?

○ Span$\{v_1, v_2\}$ is the same as Span$\{v_1 + v_2,\ -v_1 - v_2\}$.

○ If b is in Span$\{v_1, v_2\}$, then the vector equation $x_1v_1 + x_2v_2 = b$ must have exactly one solution.

● If a vector w is a linear combination of v_1 and v_2, then $-2w$ must also be a linear combination of v_1 and v_2.

○ If $v_1 \neq v_2$, then Span$\{v_1, v_2\}$ is a plane in $\mathbf{R}^n$.

(e) (2 points) Find all real values of h so that Span $\left\{\begin{pmatrix} -2 \\ 4 \end{pmatrix}, \begin{pmatrix} h \\ 12 \end{pmatrix}\right\} = \mathbf{R}^2$.

○ $h = 0$ only ○ $h = -4$ only ○ $h = -6$ only ○ $h = -12$ only

○ all real h except 0 ○ all real h except -4 ● all real h except -6

○ all real h except -12 ○ all real h ○ none of these

Problem 3 Solution.

(a) The system becomes
$$\begin{pmatrix} 3a \\ -15 \end{pmatrix} + \begin{pmatrix} 2 \\ 2b \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix},$$
so $3a + 2 = 8$ and $-15 + 2b = 1$. This gives us $a = 2$ and $b = 8$.

(b) We solve $Ax = b$ using an augmented matrix.
$$\left(A \mid b \right) = \left(\begin{array}{cc|c} 1 & -2 & -7 \\ -2 & 1 & 2 \end{array}\right) \xrightarrow{R_2 = R_2 + 2R_1} \left(\begin{array}{cc|c} 1 & -2 & -7 \\ 0 & -3 & -12 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1 + 2R_2]{R_2 = -R_2/3} \left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 4 \end{array}\right).$$
Therefore, $x = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$.

(c) The augmented matrix $\left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ -1 & -2 & -1 & -1 \end{array}\right) \to \left(\begin{array}{ccc|c} \boxed{1} & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right)$ shows that this consistent system has exactly two free variables, therefore the solution set is a plane. Since there are three variables, the solution set lives in $\mathbf{R}^3$.

(d) Statement (i) is false. Note that $v_1 + v_2$ and $-v_1 - v_2$ are scalar multiples of each other, so the most that they can span together is a line.
For example, if $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then $\text{Span}\{v_1, v_2\} = \mathbf{R}^2$.
However, $\text{Span}\{v_1 + v_2, -v_1 - v_2\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix} \right\} = \text{Span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$.

Statement (ii) is false: for example if $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$, and $b = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, then the vector equation $x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ has infinitely many solutions.

Statement (iii) is true: if $x_1 v_1 + x_2 v_2 = w$ for some real numbers x_1 and x_2, then $-2x_1 v_1 - 2x_2 v_2 = -2w$, so $-2w$ is also a linear combination of v_1 and v_2.

Statement (iv) is false and was done on a worksheet and a previous quiz. For example, if $v_1 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ vector and $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then $\text{Span}\{v_1, v_2\}$ is just a line.

(e) We need the matrix $\begin{pmatrix} -2 & h \\ 4 & 12 \end{pmatrix}$ to have a pivot in every row, so we row-reduce.
$$\begin{pmatrix} -2 & h \\ 4 & 12 \end{pmatrix} \xrightarrow{R_2 = R_2 + 2R_1} \begin{pmatrix} -4 & h \\ 0 & 12 + 2h \end{pmatrix}.$$
The bottom right entry is a pivot if and only if $12 + 2h \neq 0$, so $h \neq -6$.

4. On this page, you do not need to show work. Only your answers are graded. Parts (a) through (d) are unrelated.

(a) (2 points) Compute $\begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \end{pmatrix}$.

○ $\begin{pmatrix} -8 \\ 1 \\ 8 \end{pmatrix}$ ○ $\begin{pmatrix} -8 \\ -11 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} -10 \\ -11 \\ 8 \end{pmatrix}$ ● $\begin{pmatrix} -8 \\ -11 \\ 8 \end{pmatrix}$ ○ $\begin{pmatrix} 2 & -10 \\ -6 & -5 \\ 8 & 0 \end{pmatrix}$

(b) (2 points) Suppose a matrix A satisfies $A\begin{pmatrix} 1 \\ -9 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $A\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

Compute Av for the vector $v = \begin{pmatrix} 1 \\ -9 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 2 \end{pmatrix}$.

○ $Av = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$ ○ $Av = \begin{pmatrix} 1 \\ -7 \end{pmatrix}$ ○ $Av = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$ ● $Av = \begin{pmatrix} -3 \\ 4 \end{pmatrix}$

○ $Av = \begin{pmatrix} -9 \\ 2 \end{pmatrix}$ ○ $Av = \begin{pmatrix} 1 & -3 \\ -9 & 3 \end{pmatrix}$ ○ $Av = \begin{pmatrix} 0 & -3 \\ 1 & 3 \end{pmatrix}$

○ $Av = \begin{pmatrix} 1 & 0 \\ -9 & 6 \end{pmatrix}$ ○ not enough info to find Av.

(c) (3 points) Suppose A is a 3×4 matrix whose RREF has two pivots. Answer the following questions by filling in the appropriate bubble in each case.

i. If a vector v is a linear combination of the columns of A, then v is in:
○ $\mathbf{R}^2$ ● $\mathbf{R}^3$ ○ $\mathbf{R}^4$ ○ not enough info to determine

ii. If a vector w is a solution to $Ax = 0$, then w is in:
○ $\mathbf{R}^2$ ○ $\mathbf{R}^3$ ● $\mathbf{R}^4$ ○ not enough info to determine

iii. Which one of the following describes the set of solutions to the homogeneous equation $Ax = 0$?
○ a point ○ a line ● a plane ○ all of $\mathbf{R}^3$ ○ all of $\mathbf{R}^4$

(d) (3 points) Suppose the solution set to a matrix equation $Ax = b$ has parametric vector form $x = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ where x_2 is any real number. Which of the following are solutions to $Ax = 0$? Fill in the bubble for all that apply.

● $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ ● $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Problem 4 Solution.

(a) $\begin{pmatrix} 1 & 2 \\ -3 & 1 \\ 4 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \end{pmatrix} 2 \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} - 5 \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ -6 \\ 8 \end{pmatrix} + \begin{pmatrix} -10 \\ -5 \\ 0 \end{pmatrix} = \begin{pmatrix} -8 \\ -11 \\ 8 \end{pmatrix}$

(b) Just to make notation simpler, we let $x = \begin{pmatrix} 1 \\ -9 \end{pmatrix}$ and $y = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$, so that $v = x+3y$.

By properties of matrices:

$$Av = A(x + 3y) = Ax + 3Ay = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} -3 \\ 3 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \end{pmatrix}.$$

(c) For (i): Since A is a 3×4 matrix, each of its four columns has 3 entries, so every vector in the span of the columns of A is a vector in $\mathbf{R}^3$.

For (ii): Suppose $Aw = 0$. In order to take the product Aw, the vector w must have the same number of entries as A has columns, so w must be a vector in $\mathbf{R}^4$.

For (iii): Since A is 3×4 with 2 pivots in its RREF, there will be exactly two columns of A without a pivot, therefore there will be two free variables in the solution set to $Ax = 0$. Therefore, the solution set to $Ax = 0$ will be a plane.

(d) By the theory of section 2.4, the vector $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ is a solution to $Ax = b$, while all vectors of the form $x_2 \begin{pmatrix} 6 \\ 1 \end{pmatrix}$ are solutions to $Ax = 0$.

This means that $\begin{pmatrix} 6 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ are solutions to $Ax = 0$, but $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 0 \end{pmatrix}$ are **not** solutions to $Ax = 0$.

5. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.
For this page of the exam, consider the following linear system of equations in the variables $x_1,\ x_2,\ x_3,\ x_4$:

$$x_1 + 3x_3 - 3x_4 = 3$$
$$2x_1 + x_2 + 6x_3 - 4x_4 = 5$$
$$-2x_1 - 2x_2 - 5x_3 - x_4 = -5.$$

(a) (4 points) Write the system in the form of an augmented matrix, and put the augmented matrix in reduced row echelon form.

Solution: We box the pivots below.

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -3 & 3 \\ 2 & 1 & 6 & -4 & 5 \\ -2 & -2 & -5 & -1 & -5 \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2-2R_1} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -3 & 3 \\ 0 & \boxed{1} & 0 & 2 & -1 \\ 0 & -2 & 1 & -7 & 1 \end{array}\right)$$

$$\xrightarrow{R_3=R_3+2R_2} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 3 & -3 & 3 \\ 0 & \boxed{1} & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & -3 & -1 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-3R_3} \left(\begin{array}{cccc|c} \boxed{1} & 0 & 0 & 6 & 6 \\ 0 & \boxed{1} & 0 & 2 & -1 \\ 0 & 0 & \boxed{1} & -3 & -1 \end{array}\right)$$

(b) (4 pts) The system is consistent. Write its solution set in parametric **vector** form.

Solution: We see $x_1 + 6x_4 = 6$, $x_2 + 2x_4 = -1$, $x_3 - 3x_4 = -1$, and x_4 is free, so:

$$x_1 = 6 - 6x_4, \quad x_2 = -1 - 2x_4, \quad x_3 = -1 + 3x_4, \quad x_4 = x_4 \ (x_4 \text{ real}).$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 - 6x_4 \\ -1 - 2x_4 \\ -1 + 3x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -6x_4 \\ -2x_4 \\ 3x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 6 \\ -1 \\ -1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -6 \\ -2 \\ 3 \\ 1 \end{pmatrix}$$

(c) (2 points) Write two different solutions u and v to the linear system in the space provided below. Only your answer is graded on this part, so please check by hand that your answer is correct, and if it is not correct then check your work above!

$$u = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix} \qquad\qquad v = \begin{pmatrix} \\ \\ \\ \\ \end{pmatrix}$$

Solution: One solution to the system is $\begin{pmatrix} 6 \\ -1 \\ -1 \\ 0 \end{pmatrix}$, and to get more solutions to the system we can add any multiple of the **homogeneous** solution $\begin{pmatrix} -6 \\ -2 \\ 3 \\ 1 \end{pmatrix}$, for example:

$$\begin{pmatrix} 6 \\ -1 \\ -1 \\ 0 \end{pmatrix} + \begin{pmatrix} -6 \\ -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -3 \\ 2 \\ 1 \end{pmatrix},$$

$$\begin{pmatrix} 6 \\ -1 \\ -1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} -6 \\ -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ -5 \\ 5 \\ 2 \end{pmatrix},$$

$$\begin{pmatrix} 6 \\ -1 \\ -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -6 \\ -2 \\ 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 1 \\ -4 \\ -1 \end{pmatrix},$$

etc.

6. Free response. Show your work unless otherwise indicated! A correct answer without appropriate work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (5 points) Consider the vector equation

$$x_1 \begin{pmatrix} 1 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} -3 \\ -12 \end{pmatrix} = \begin{pmatrix} -2 \\ -8 \end{pmatrix}.$$

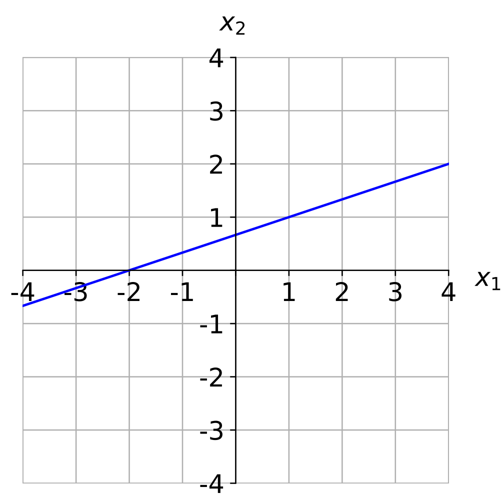
i. Find the set of solutions to the vector equation, and write it in parametric form.

Solution: The matrix $\left(\begin{array}{cc|c} 1 & -3 & -2 \\ 4 & -12 & -8 \end{array}\right)$ row-reduces in one step to $\left(\begin{array}{cc|c} 1 & -3 & -2 \\ 0 & 0 & 0 \end{array}\right)$, so $x_1 - 3x_2 = -2$ and x_2 is free. Therefore,

$$x_1 = -2 + 3x_2, \qquad x_2 = x_2 \ (x_2 \text{ real}).$$

ii. Using your answer from part (i), draw the solution set for the vector equation below very carefully. You do not need to show your work for this part.

Solution: There is one free variable, so the solution set is a line. We could use the parametric form to get points $(-2, 0)$ (when $x_2 = 0$) and $(1, 1)$ (when $x_2 = 1$) to get the line. Alternatively, we could use the parametric v.f. $\begin{pmatrix} -2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ to draw the line through $(-2, 0)$ parallel to the span of $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.



(b) (5 points) Find all values of h and k so that the system below has infinitely many solutions. Clearly mark your answers for h and k.

$$2x - 8y = k$$

$$6x - hy = 7.$$

Solution: This is as standard as it gets.

$$\left(\begin{array}{cc|c} 2 & -8 & k \\ 6 & -h & 7 \end{array}\right) \xrightarrow{R_2 = R_2 - 3R_1} \left(\begin{array}{cc|c} 2 & -8 & k \\ 0 & -h+24 & 7-3k \end{array}\right).$$

To get infinitely many solutions, the second row must not have any pivot at all, so $-h + 24 = 0$ and $7 - 3k = 0$. Therefore, $h = 24$ and $k = 7/3$.

This question is the same as asking to find h and k so that the two equations $2x - 8y = k$ and $6x - hy = 7$ define the same line. Multiplying the first equation by 3 and setting it equal to the second would give us $h = 24$ and $k = 7/3$.

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work will receive little or no credit.

(a) (2 points) Write a matrix A with the property that $Ax = b$ is consistent if and only if b is in Span $\left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$. You do not need to show work on this part.

We need to write a matrix A so that the span of its columns is exactly the plane given above. Note that for everything in the span of those two vectors, the second entry is -1 times the first, so each column of A must satisfy that property. Also, we need to have a vector with a nonzero third entry.

For this, we can simply take A to be $A = \begin{pmatrix} 1 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix}$.

There are many other possibilities, for example we could create a third column that is a linear combination of the first two (this will not change the overall span), such as

$$A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We could even take a third column of all zeros:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We could have the first and second column each be a (nonzero) multiple of the vectors we were given, or swap them, or both, etc.

$$A = \begin{pmatrix} -1 & 0 \\ 1 & 0 \\ 0 & 5 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 \\ 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 10 \\ 0 & -10 \\ 4 & 0 \end{pmatrix}, \qquad \text{etc.}$$

(b) (3 points) Consider the linear system of system of equations in x_1, x_2, x_3, and x_4 whose augmented matrix is given below:

$$\left(\begin{array}{cccc|c} 1 & -9 & 0 & 1 & 2 \\ 0 & 1 & 2 & -1 & -1 \end{array}\right).$$

Write the set of solutions to the system in parametric form. Indicate which variables (if any) are free.

Solution: Adding $9R_2$ to R_1 puts the matrix in RREF:

$$\left(\begin{array}{cccc|c} \boxed{1} & 0 & 18 & -8 & -7 \\ 0 & \boxed{1} & 2 & -1 & -1 \end{array}\right).$$

Thus, $x_1 + 18x_3 - 8x_4 = -7$ and $x_2 + 2x_3 - x_4 = -1$, where x_3 and x_4 are free.

$x_1 = -7 - 18x_3 + 8x_4, \quad x_2 = -1 - 2x_3 + x_4, \quad x_3 = x_3 \ (x_3 \text{ real}), \quad x_4 = x_4 \ (x_4 \text{ real}).$

(c) (5 points) Let $A = \begin{pmatrix} 1 & -3 \\ 3 & 1 \\ 4 & -10 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ h \\ -4 \end{pmatrix}$.

Find all real values of h so that the matrix equation $Ax = b$ is **inconsistent**.

Solution: We row-reduce $\left(\begin{array}{c|c} A & b \end{array}\right)$ to find when there is a pivot in the rightmost column of

$$\left(\begin{array}{cc|c} 1 & -3 & 1 \\ 3 & 1 & h \\ 4 & -10 & -4 \end{array}\right) \xrightarrow[R_3=R_3-4R_1]{R_2=R_2-3R_1} \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 10 & h-3 \\ 0 & 2 & -8 \end{array}\right) \xrightarrow[\text{then } R_2=R_2/2]{R_2\longleftrightarrow R_3} \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -4 \\ 0 & 10 & h-3 \end{array}\right)$$

$$\xrightarrow{R_3=R_3-10R_2} \left(\begin{array}{cc|c} 1 & -3 & 1 \\ 0 & 1 & -4 \\ 0 & 0 & h+37 \end{array}\right)$$

The system is inconsistent precisely when there is a pivot in the rightmost column, which is when $h + 37 \neq 0$. Therefore, $h \neq -37$.

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