

# Chapter 4

## Determinants

# Section 4.1

Determinants: Definition

**Recall:** This course is about learning to:

- ▶ Solve the matrix equation  $Ax = b$   
We've said most of what we'll say about this topic now.
- ▶ Solve the matrix equation  $Ax = \lambda x$  (eigenvalue problem)  
We are now aiming at this.
- ▶ Almost solve the equation  $Ax = b$   
This will happen later.

The next topic is *determinants*.

This is a completely magical function that takes a square matrix and gives you a number.

It is a very complicated function—the formula for the determinant of a  $10 \times 10$  matrix has 3,628,800 summands—so instead of writing down the formula, we'll give other ways to compute it.

Today is mostly about the *theory* of the determinant; in the next lecture we will focus on *computation*.

# A Definition of Determinant

## Definition

The **determinant** is a function

determinants are only for square matrices!

$$\det: \{n \times n \text{ matrices}\} \rightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by  $c$ , the determinant is multiplied by  $c$ .
3. If you swap two rows of a matrix, the determinant is multiplied by  $-1$ .
4.  $\det(I_n) = 1$ .

Example:

# A Definition of Determinant

## Definition

The **determinant** is a function

determinants are only for square matrices!

$$\det: \{n \times n \text{ matrices}\} \rightarrow \mathbf{R}$$

with the following properties:

1. If you do a row replacement on a matrix, the determinant doesn't change.
2. If you scale a row by  $c$ , the determinant is multiplied by  $c$ .
3. If you swap two rows of a matrix, the determinant is multiplied by  $-1$ .
4.  $\det(I_n) = 1$ .

This is a *definition* because it tells you how to compute the determinant: row reduce!

It's not at all obvious that you get the same determinant if you row reduce in two different ways, but this is magically true!

## Special Cases

### Special Case 1

If  $A$  has a zero row, then  $\det(A) = 0$ .

Why?

## Special Case 2

If  $A$  is upper-triangular, then the determinant is the product of the diagonal entries:

$$\det \begin{pmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{pmatrix} = abc.$$

**Upper-triangular** means the only nonzero entries are on or above the diagonal.

Why?

# Computing Determinants

## Method 1

### Theorem

Let  $A$  be a square matrix. Suppose you do some number of row operations on  $A$  to get a matrix  $B$  in row echelon form. Then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})},$$

where  $r$  is the number of row swaps.

Why?

### Remark

This is generally the fastest way to compute a determinant of a large matrix, either by hand or by computer.

Row reduction is  $O(n^3)$ ; cofactor expansion (next time) is  $O(n!) \sim O(n^n \sqrt{n})$ .

This is important in real life, when you're usually working with matrices with a gazillion columns.



# Computing Determinants

## Example

# Computing Determinants

## $2 \times 2$ Example

Let's compute the determinant of  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ , a general  $2 \times 2$  matrix.


- ▶ If  $a = 0$ , then

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

- ▶ Otherwise,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

In both cases, the determinant magically turns out to be


$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$



## Theorem

A square matrix  $A$  is invertible if and only if  $\det(A)$  is nonzero.

## Why?

- ▶ If  $A$  is invertible, then its reduced row echelon form is the identity matrix, which has determinant equal to 1.
- ▶ If  $A$  is not invertible, then its reduced row echelon form has a zero row, hence has zero determinant.
- ▶ Doing row operations doesn't change whether the determinant is zero.

## Theorem

If  $A$  and  $B$  are two  $n \times n$  matrices, then

$$\det(AB) = \det(A) \cdot \det(B).$$

Why?

## Theorem

If  $A$  is invertible, then  $\det(A^{-1}) = \frac{1}{\det(A)}.$

Why?

# Transposes

## Review

**Recall:** The **transpose** of an  $m \times n$  matrix  $A$  is the  $n \times m$  matrix  $A^T$  whose rows are the columns of  $A$ . In other words, the  $ij$  entry of  $A^T$  is  $a_{ji}$ .

$$\begin{matrix} & A & & A^T \\ \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} & \rightsquigarrow & \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \\ a_{13} & a_{23} \end{pmatrix} \end{matrix}$$

# Determinants and Transposes

## Theorem

If  $A$  is a square matrix, then

$$\det(A) = \det(A^T),$$

where  $A^T$  is the transpose of  $A$ .

**Example:**  $\det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = \det \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}.$

As a consequence,  $\det$  behaves the same way with respect to *column* operations as row operations.

**Corollary**  an immediate consequence of a theorem

If  $A$  has a zero column, then  $\det(A) = 0$ .

## Corollary

The determinant of a *lower*-triangular matrix is the product of the diagonal entries.

(The transpose of a lower-triangular matrix is upper-triangular.)

## Section 4.3

### Determinants and Volumes

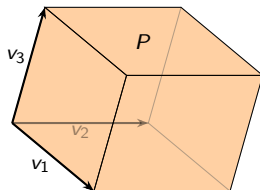
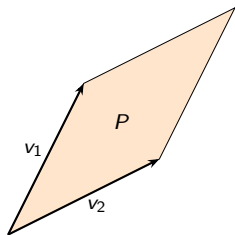


# Determinants and Volumes

Now we discuss a completely different description of (the absolute value of) the determinant, in terms of volumes.

This is a crucial component of the change-of-variables formula in multivariable calculus.

The columns  $v_1, v_2, \dots, v_n$  of an  $n \times n$  matrix  $A$  give you  $n$  vectors in  $\mathbf{R}^n$ . These determine a **parallelepiped**  $P$ .



## Theorem

Let  $A$  be an  $n \times n$  matrix with columns  $v_1, v_2, \dots, v_n$ , and let  $P$  be the parallelepiped determined by  $A$ . Then

$$(\text{volume of } P) = |\det(A)|.$$

## Theorem

Let  $A$  be an  $n \times n$  matrix with columns  $v_1, v_2, \dots, v_n$ , and let  $P$  be the parallelepiped determined by  $A$ . Then

$$(\text{volume of } P) = |\det(A)|.$$

**Sanity check:** the volume of  $P$  is zero  $\iff$  the columns are *linearly dependent* ( $P$  is “flat”)  $\iff$  the matrix  $A$  is not invertible.

**Why is the theorem true?** You only have to check that the volume behaves the same way under row operations as  $|\det|$  does.

Note that the volume of the unit cube (the parallelepiped defined by the identity matrix) is 1.

# Determinants and Volumes

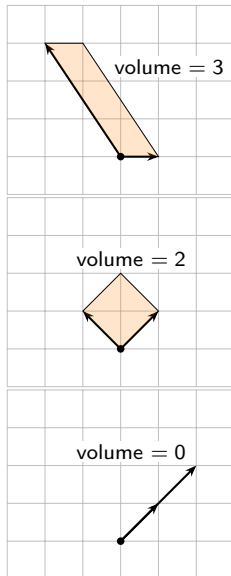
Examples in  $\mathbb{R}^2$

$$\det \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix} = 3$$

$$\det \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} = -2$$

(Should the volume really be  $-2$ ?)

$$\det \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} = 0$$



# Determinants and Volumes

## Theorem

Let  $A$  be an  $n \times n$  matrix with columns  $v_1, v_2, \dots, v_n$ , and let  $P$  be the parallelepiped determined by  $A$ . Then

$$(\text{volume of } P) = |\det(A)|.$$

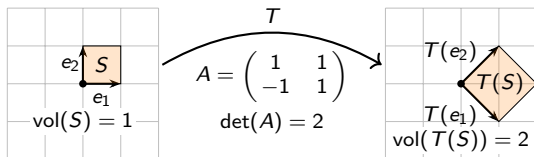
This is even true for curvy shapes, in the following sense.

## Theorem

Let  $A$  be an  $n \times n$  matrix, and let  $T(x) = Ax$ . If  $S$  is any region in  $\mathbf{R}^n$ , then

$$(\text{volume of } T(S)) = |\det(A)| (\text{volume of } S).$$

If  $S$  is the unit cube, then  $T(S)$  is the parallelepiped defined by the columns of  $A$ , since the columns of  $A$  are  $T(e_1), T(e_2), \dots, T(e_n)$ . In this case, the second theorem is the same as the first.



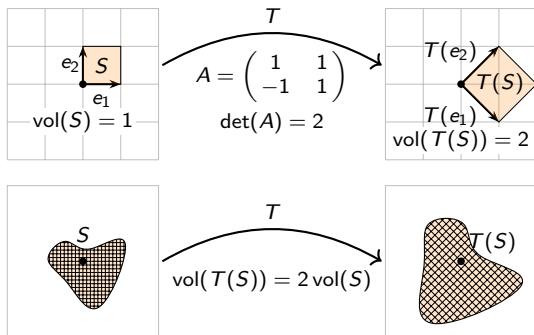
# Determinants and Volumes

## Theorem

Let  $A$  be an  $n \times n$  matrix, and let  $T(x) = Ax$ . If  $S$  is any region in  $\mathbf{R}^n$ , then

$$(\text{volume of } T(S)) = |\det(A)| (\text{volume of } S).$$

For curvy shapes, you break  $S$  up into a bunch of tiny cubes. Each one is scaled by  $|\det(A)|$ ; then you use *calculus* to reduce to the previous situation!



# Determinants and Volumes

## Example

### Theorem

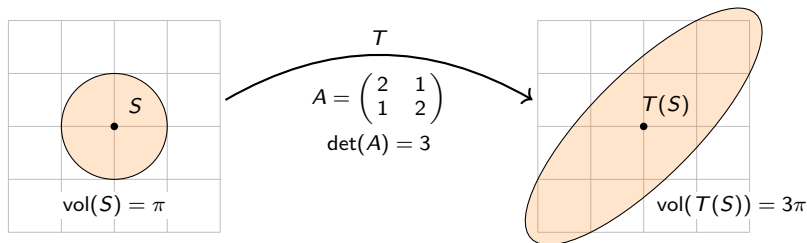
Let  $A$  be an  $n \times n$  matrix, and let  $T(x) = Ax$ . If  $S$  is any region in  $\mathbf{R}^n$ , then

$$(\text{volume of } T(S)) = |\det(A)| (\text{volume of } S).$$

**Example:** Let  $S$  be the unit disk in  $\mathbf{R}^2$ , and let  $T(x) = Ax$  for

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

Note that  $\det(A) = 3$ .



## Magical Properties of the Determinant

you really have to know these

1. There is one and only one function  $\det: \{\text{square matrices}\} \rightarrow \mathbf{R}$  satisfying the properties (1)–(4) on the second slide.
2.  $A$  is invertible if and only if  $\det(A) \neq 0$ .
3. The determinant of an upper- or lower-triangular matrix is the product of the diagonal entries.
4. If we row reduce  $A$  to row echelon form  $B$  using  $r$  swaps, then

$$\det(A) = (-1)^r \frac{(\text{product of the diagonal entries of } B)}{(\text{product of the scaling factors})}.$$

5.  $\det(AB) = \det(A)\det(B)$  and  $\det(A^{-1}) = \det(A)^{-1}$ .
6.  $\det(A) = \det(A^T)$ .
7.  $|\det(A)|$  is the volume of the parallelepiped defined by the columns of  $A$ .
8. If  $A$  is an  $n \times n$  matrix with transformation  $T(x) = Ax$ , and  $S$  is a subset of  $\mathbf{R}^n$ , then the volume of  $T(S)$  is  $|\det(A)|$  times the volume of  $S$ . (Even for curvy shapes  $S$ .)