Section 2.6

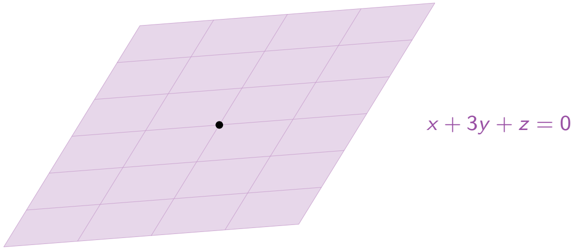
Subspaces

Motivation

Today we will discuss **subspaces** of $\mathbf{R}^n$.

A subspace turns out to be the same as a span, except we don't know *which* vectors it's the span of.

This arises naturally when you have, say, a plane through the origin in $\mathbf{R}^3$ which is *not* defined (a priori) as a span, but you still want to say something about it.



Definition of Subspace

Definition

A **subspace** of $\mathbf{R}^n$ is a subset V of $\mathbf{R}^n$ satisfying:

1. The zero vector is in V. "not empty"
2. If u and v are in V, then $u + v$ is also in V. "closed under addition"
3. If u is in V and c is in $\mathbf{R}$, then cu is in V. "closed under $\times$ scalars"

Fast-forward

Every subspace is a span, and every span is a subspace.

A subspace is a span of some vectors, but you haven't computed what those vectors are yet.

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What does this mean?

- If v is in V, then all scalar multiples of v are in V by (3). In other words, the line through any nonzero vector in V is also in V.
- If u, v are in V, then cu and dv are in V for any scalars c, d by (3). So $cu + dv$ is in V by (2). So Span$\{u, v\}$ is contained in V.
- Likewise, if $v_1, v_2, \ldots, v_n$ are all in V, then Span$\{v_1, v_2, \ldots, v_n\}$ is contained in V: a subspace contains the span of any set of vectors in it.

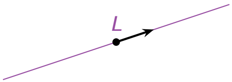
If you pick enough vectors in V, eventually their span will fill up V, so:

> A subspace is a span of some set of vectors in it.

Examples

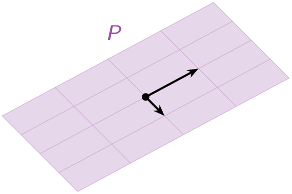
Example

A line L through the origin is a subspace: L contains zero and is easily seen to be closed under addition and scalar multiplication.



Example

A plane P through the origin is a subspace: P contains zero; the sum of two vectors in P is also in P; and any scalar multiple of a vector in P is also in P.



Example

All of $\mathbf{R}^n$: this contains 0, and is closed under addition and scalar multiplication.

Example

The subset $\{0\}$: this subspace contains only one vector.

Note these are all pictures of spans! (Line, plane, space, etc.)

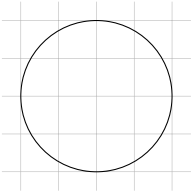
A **subset** of $\mathbf{R}^n$ is any collection of vectors in $\mathbf{R}^n$ whatsoever. For example, the unit circle

$$C = \{(x, y) \text{ in } \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

is a subset of $\mathbf{R}^2$, but it is not a subspace.

All of the following non-examples on the next slide are still subsets.

A **subspace** is a special kind of subset, that satisfies the three defining properties.

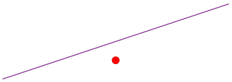


Subset: *yes*
Subspace: *no*

Non-Examples

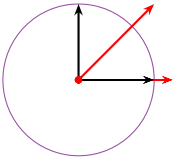
Non-Example

A line L (or any other set) that doesn't contain the origin is not a subspace. Fails: 1.



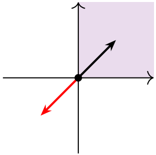
Non-Example

A circle C is not a subspace. Fails: 1,2,3. Think: a circle isn't a "linear space."



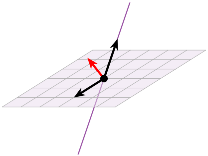
Non-Example

The first quadrant in $\mathbf{R}^2$ is not a subspace. Fails: 3 only.



Non-Example

A line union a plane in $\mathbf{R}^3$ is not a subspace. Fails: 2 only.



Subspaces are Spans, and Spans are Subspaces

Theorem

Any $\text{Span}\{v_1, v_2, \ldots, v_p\}$ is a subspace.

!!!

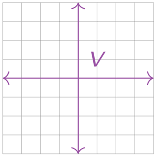
Every subspace is a span, and every span is a subspace.

Definition

If $V = \text{Span}\{v_1, v_2, \ldots, v_p\}$, we say that V is the subspace **generated by** or **spanned by** the vectors $v_1, v_2, \ldots, v_p$. We call $\{v_1, v_2, \ldots, v_p\}$ a **spanning set** for V.

Poll

Let $V = \left\{ \begin{pmatrix} a \\ b \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; ab = 0 \right\}$. Let's check if V is a subspace or not.



We conclude that V is *not* a subspace. A picture is above. (It doesn't look like a span.)

Column Space and Null Space

An $m \times n$ matrix A naturally gives rise to *two* subspaces.

Definition

- The **column space** of A is the subspace of $\mathbf{R}^m$ spanned by the columns of A. It is written $\operatorname{Col} A$.
- The **null space** of A is the set of all solutions of the homogeneous equation $Ax = 0$:

  $$\operatorname{Nul} A = \{x \text{ in } \mathbf{R}^n \mid Ax = 0\}.$$

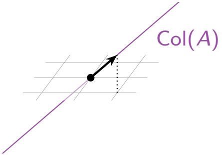
  This is a subspace of $\mathbf{R}^n$.

The column space is defined as a span, so we know it is a subspace.

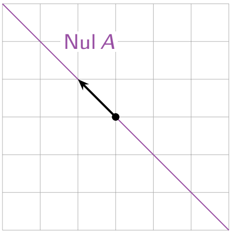
Check that the null space is a subspace:

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Let's compute the column space:



Let's compute the null space:



The Null Space is a Span

The column space of a matrix A is defined to be a span (of the columns).

The null space is defined to be the solution set to $Ax = 0$. It is a subspace, so it is a span.

Question

How to find vectors that span the null space?

Answer: Parametric vector form! We know that the solution set to $Ax = 0$ has a parametric form that looks like

$$x_3 \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix}$$

if, say, x_3 and x_4 are the free variables. So

$$\text{Nul}\, A = \text{Span}\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Refer back to the slides for §2.4 (Solution Sets).

Note: It is much easier to define the null space first as a subspace, then find spanning vectors *later*, if we need them. This is one reason subspaces are so useful.

- A **subspace** is the same as a span of some number of vectors, but we haven't computed the vectors yet.
- To any matrix is associated two subspaces, the **column space** and the **null space**:

$$\mathrm{Col}\,A = \text{the span of the columns of } A$$

$$\mathrm{Nul}\,A = \text{the solution set of } Ax = 0.$$

How do you check if a subset is a subspace?

- Is it a span? Can it be written as a span?
- Can it be written as the column space of a matrix?
- Can it be written as the null space of a matrix?
- Is it all of $\mathbf{R}^n$ or the zero subspace $\{0\}$?
- Can it be written as a type of subspace that we'll learn about later (eigenspaces, ...)?

If so, then it's automatically a subspace.

If all else fails:

- Can you verify directly that it satisfies the three defining properties?