Chapter 1

Systems of Linear Equations: Algebra

Section 1.1

Systems of Linear Equations

Line, Plane, Space, ...

Recall that $\mathbf{R}$ denotes the collection of all real numbers, i.e. the number line. It contains numbers like $0, -1, \pi, \frac{3}{2}, \ldots$

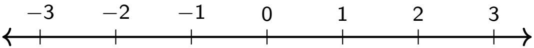
Definition

Let n be a positive whole number. We define

$$\mathbf{R}^n = \text{all ordered } n\text{-tuples of real numbers } (x_1, x_2, x_3, \ldots, x_n).$$

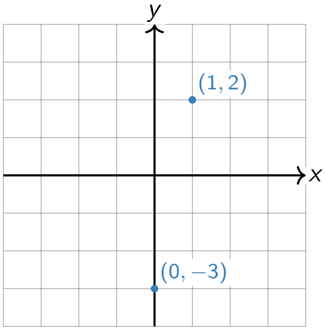
Example

When $n = 1$, we just get $\mathbf{R}$ back: $\mathbf{R}^1 = \mathbf{R}$. Geometrically, this is the *number line*.



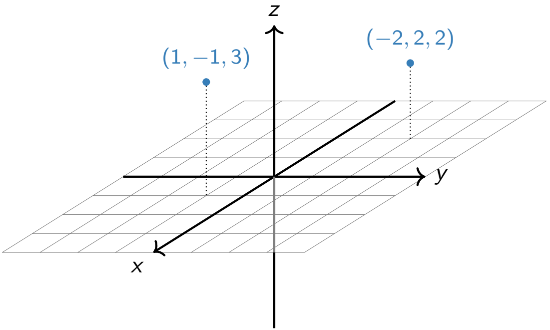
Example

When $n = 2$, we can think of $\mathbf{R}^2$ as the *plane*. This is because every point on the plane can be represented by an ordered pair of real numbers, namely, its x- and y-coordinates.



Example

When $n = 3$, we can think of $\mathbf{R}^3$ as the *space* we (appear to) live in. This is because every point in space can be represented by an ordered triple of real numbers, namely, its x-, y-, and z-coordinates.



Line, Plane, Space, . . .
Continued

So what is $\mathbf{R}^4$? or $\mathbf{R}^5$? or $\mathbf{R}^n$?

. . . go back to the *definition*: ordered n-tuples of real numbers

$$(x_1, x_2, x_3, \ldots, x_n).$$

They're still "geometric" spaces, in the sense that our intuition for $\mathbf{R}^2$ and $\mathbf{R}^3$ sometimes extends to $\mathbf{R}^n$, but they're harder to visualize.

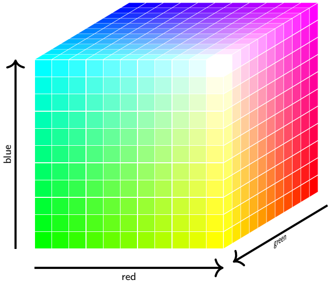
We'll make definitions and state theorems that apply to any $\mathbf{R}^n$, but we'll only draw pictures for $\mathbf{R}^2$ and $\mathbf{R}^3$.

The power of using these spaces is the ability to use elements of $\mathbf{R}^n$ to *label* various objects of interest, like solutions to systems of equations.

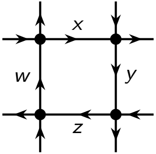
Labeling with $\mathbf{R}^n$

Example

All colors you can see can be described by three quantities: the amount of red, green, and blue light in that color. Therefore, we can use the elements of $\mathbf{R}^3$ to *label* all colors: the point $(.2, .4, .9)$ labels the color with 20% red, 40% green, and 90% blue.



Last time we could have used $\mathbf{R}^4$ to *label* the amount of traffic (x, y, z, w) passing through four streets.

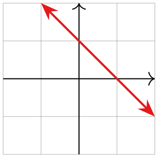


For instance the point $(100, 20, 30, 150)$ corresponds to a situation where 100 cars per hour drive on road x, 20 cars per hour drive on road y, etc.

One Linear Equation

What does the solution set of a linear equation look like?

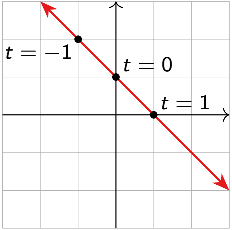
$x + y = 1 \rightsquigarrow$ a line in the plane: $y = 1 - x$
This is called the **implicit equation** of the line.



We can write the same line in **parametric form** in $\mathbf{R}^2$:

$$(x,\, y) = (t,\, 1 - t) \qquad t \text{ in } \mathbf{R}.$$

This means that every point on the line has the form $(t, 1 - t)$ for some real number t. Note we are using $\mathbf{R}$ to *label* the points on a line in $\mathbf{R}^2$.

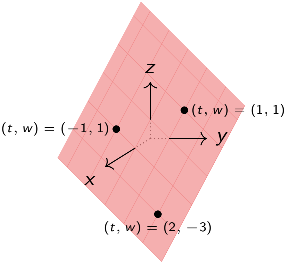


Aside

What is a line? A ray that is *straight* and infinite in both directions.

What does the solution set of a linear equation look like?

$x + y + z = 1 \rightsquigarrow$ a plane in space:
This is the **implicit equation** of the plane.



[interactive]

Does this plane have a **parametric form**?

$$(x, y, z) = (1 - t - w,\ t,\ w) \qquad t, w \text{ in } \mathbf{R}.$$

Note we are using $\mathbf{R}^2$ to *label* the points on a plane in $\mathbf{R}^3$.

Aside

What is a plane? A flat sheet of paper that's infinite in all directions.

One Linear Equation
Continued

What does the solution set of a linear equation look like?

$x + y + z + w = 1 \rightsquigarrow$ a "3-plane" in "4-space"... [not pictured here]

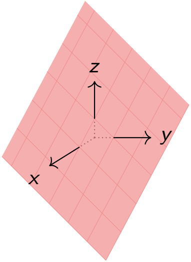
Poll

Everybody get out your gadgets!

Poll

Is the plane from the previous example equal to $\mathbf{R}^2$?

A. Yes **B.** No



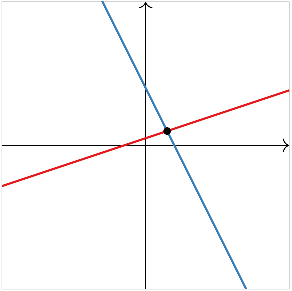
No! Every point on this plane is in $\mathbf{R}^3$: that means it has three coordinates. For instance, $(1, 0, 0)$. Every point in $\mathbf{R}^2$ has two coordinates. But we can *label* the points on the plane by $\mathbf{R}^2$.

Systems of Linear Equations

What does the solution set of a *system* of more than one linear equation look like?

$$x - 3y = -3$$
$$2x + y = 8$$

. . . is the *intersection* of two lines, which is a *point* in this case.



In general it's an intersection of lines, planes, etc.

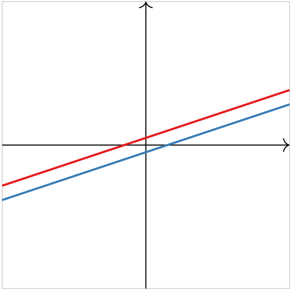
[two planes intersecting]

Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$x - 3y = 3$$

has no solution: the lines are *parallel*.



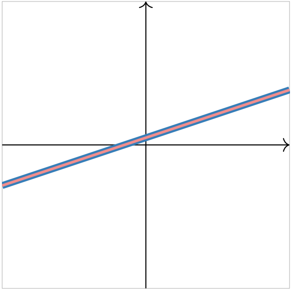
A system of equations with no solutions is called **inconsistent**.

Kinds of Solution Sets

In what other ways can two lines intersect?

$$x - 3y = -3$$
$$2x - 6y = -6$$

has infinitely many solutions:
they are the *same line*.



Note that multiplying an equation by a nonzero number gives the *same solution set*. In other words, they are *equivalent* (systems of) equations.

Summary

- $\mathbf{R}^n$ is the set of ordered lists of n numbers.
- $\mathbf{R}^n$ can be used to label geometric objects, like $\mathbf{R}^2$ can label points on a plane.
- The solutions of a system equations look like an intersection of lines, planes, etc.
- Finding all the solutions of a system of equations means finding a **parametric form**: a labeling by some $\mathbf{R}^n$.