Math 1553 Worksheet §3.4-3.6

Solutions

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

a) If A and B are $n \times n$ matrices and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

b) If A is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each b in $\mathbf{R}^n$, then the solution is *unique* for each b in $\mathbf{R}^n$.

c) If A is a 3×4 matrix and B is a 4×2 matrix, then the linear transformation Z defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

d) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

e) If A, B, and C are nonzero 2×2 matrices satisfying $BA = CA$, then $B = C$.

Solution.

a) False. $(AB)^{-1} = B^{-1}A^{-1}$.

b) True. The first part says the transformation $T(x) = Ax$ is onto. Since A is $n \times n$, then it has n pivots. This is the same as saying A is invertible, and there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each b in $\mathbf{R}^n$.

Even in the case when $A + B$ is invertible, it still might not be true that $(A + B)^{-1} = A^{-1} + B^{-1}$. For example, $(I_2 + I_2)^{-1} = (2I_2)^{-1} = \frac{1}{2}I_2$, whereas $(I_2)^{-1} + (I_2)^{-1} = I_2 + I_2 = 2I_2$.

c) False. In order for Bx to make sense, x must be in $\mathbf{R}^2$, and so Bx is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of Z is $\mathbf{R}^2$ and the codomain of Z is $\mathbf{R}^3$.

d) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

e) False. Take $A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $BA = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $BC = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, but $B \neq C$.

2. A is $m \times n$ matrix, B is $n \times m$ matrix. Select all correct answers from the box. It is possible to have more than one correct answer.

a) Suppose x is in $\mathbf{R}^m$. Then ABx *must be* in:
$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

b) Suppose x in $\mathbf{R}^n$. Then BAx *must be* in:
$\boxed{\text{Col}(A), \quad \text{Nul}(A), \quad \text{Col}(B), \quad \text{Nul}(B)}$

c) If $m > n$, then columns of AB could be linearly $\boxed{independent, \quad dependent}$

d) If $m > n$, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

e) If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of BA could be linearly $\boxed{independent, \quad dependent}$

Solution.

Recall, AB can be computed as A multiplying every column of B. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

a) $\boxed{\text{Col}(A)}$. Denote $w := Bx$, which is a vector in $\mathbf{R}^n$. $ABx = A(Bx)$ is multiplying A with w which will end up with "linear combination of columns of A". So ABx is in Col(A).

b) $\boxed{\text{Col}(B)}$. Similarly, $BAx = B(Ax)$ is multiplying B with Ax, a vector in R^m, which will end up with "linear combination of columns of B". So BAx is in Col(B).

c) $\boxed{dependent}$. Since $m > n$ means A matrix can have at most n pivots. So $dim(\text{Col}(A)) \leq n$. Notice from first question we know $\text{Col}(AB) \subset \text{Col}(A)$ which has dimension at most n. That means AB can have at most n pivots. But AB is $m \times m$ matrix, then columns of AB must be dependent.

d) $\boxed{independent, dependent}$. Both are possible. Since $m > n$ means B matrix can have at most n pivots. then $\text{Col}(BA) \subset \text{Col}(B)$ means BA can have at most n pivots. Since BA is $n \times n$ matrix, then the columns of BA will be linearly independent when there are n pivots or linearly dependent when there are less than n pivots. Here are two examples.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

e) $\boxed{dependent}$. From the second example above, BA has dependent columns, we know "dependent" is one possible answer. Now to see if "independent" is also possible, we need to find out if BA could have n pivots.

Since $Ax = 0$ has nontrivial solution say x^*, then x^* is also a nontrivial solution of $BAx = 0$. That means BA has free variables, and it can not have n pivots. So columns of BA must be linearly dependent.

To summarize what we are actually study here, there are several relations between these subspaces.

$$\mathrm{Col}(AB) \subset \mathrm{Col}(A);$$
$$\mathrm{Col}(BA) \subset \mathrm{Col}(B);$$
$$\mathrm{Nul}(A) \subset \mathrm{Nul}(BA);$$
$$\mathrm{Nul}(B) \subset \mathrm{Nul}(AB);$$

3. Consider the following linear transformations:

$$\begin{aligned} T: \mathbf{R}^3 &\longrightarrow \mathbf{R}^2 & &T \text{ projects onto the } xy\text{-plane, forgetting the } z\text{-coordinate} \\ U: \mathbf{R}^2 &\longrightarrow \mathbf{R}^2 & &U \text{ rotates clockwise by } 90^\circ \\ V: \mathbf{R}^2 &\longrightarrow \mathbf{R}^2 & &V \text{ scales the } x\text{-direction by a factor of 2.} \end{aligned}$$

Let A, B, C be the matrices for T, U, V, respectively.

a) Write A, B, and C.

b) Compute the matrix for $U \circ V \circ T$.

c) Describe U^{-1} and V^{-1}, and compute their matrices.

Solution.

a) We plug in the unit coordinate vectors:

$$\begin{aligned} T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad &\Longrightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ U(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad &\Longrightarrow \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \\ V(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad V(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad &\Longrightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

b) $BCA = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$.

c) Intuitively, if we wish to “undo" U, we can imagine that $\begin{pmatrix} x \\ y \end{pmatrix}$. To do this, we need to rotate it 90° *counterclockwise*. Therefore, U^{-1} is counterclockwise rotation by 90°.

Similarly, to undo the transformation V that scales the x-direction by 2, we need to scale the x-direction by $1/2$, so V^{-1} scales the x-direction by a factor of $1/2$.

Their matrices are, respectively,

$$B^{-1} = \frac{1}{0 \cdot 0 - (-1) \cdot 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$C^{-1} = \frac{1}{2 \cdot 1 - 0 \cdot 0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$