Math 1553 Worksheet §2.5, 2.6, 2.7, 2.9, 3.1

Solutions

1. If the statement is always true, circle TRUE. Otherwise, circle FALSE. Justify your answer.

a) Suppose $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $A\begin{pmatrix}-3\\2\\7\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$. Must v_1, v_2, v_3 be linearly dependent? If true, write a linear dependence relation for the vectors.
TRUE **FALSE**

b) If b is in Col(A), then so is $5b$. **TRUE** **FALSE**

c) In the following, A is an $m \times n$ matrix.

(1) **TRUE** **FALSE** If A has linearly dependent columns, then $m < n$.

(2) **TRUE** **FALSE** If A has linearly independent columns, then $Ax = b$ must have at least one solution for each b in $\mathbf{R}^m$.

(3) **TRUE** **FALSE** If b is a vector in $\mathbf{R}^m$ and $Ax = b$ has exactly one solution, then $m \geq n$.

Solution.

a) **TRUE**. By definition of matrix multiplication, $-3v_1+2v_2+7v_3 = 0$, so $\{v_1, v_2, v_3\}$ is linearly dependent and the equation gives a linear dependence relation.

b) **TRUE**. Let v be a solution to $Ax = b$, so $Av = b$. Then $A(5v) = 5Av = 5b$.

c) (1) **FALSE** For example $A = \begin{pmatrix}1 & 0\\0 & 0\end{pmatrix}$.

(Note that even though this part was false, there is a very similar-sounding statement that is true: if $m < n$, A must have linearly dependent columns.)

(2) **FALSE** For example $A = \begin{pmatrix}1\\0\end{pmatrix}$, $b = \begin{pmatrix}0\\1\end{pmatrix}$. There is no solution for $Ax = b$.

(Note, however: if A has linearly independent columns, then the system $Ax = 0$ has no free variables, so $Ax = b$ is either inconsistent or has a unique solution.)

(3) **TRUE** If $Ax = b$ has a unique solution, then since it is a translation of the solution set to $Ax = 0$, this means that $Ax = 0$ has only the trivial solution (no free variables). Thus, A has a pivot in every column, which is impossible if $m < n$ (i.e. impossible if A has more columns than rows), so $m \geq n$.

2. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×10 matrix with 2 pivots, then $\dim(\mathrm{Nul} A) = 8$ and $\mathrm{rank}(A) = 2$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the transformation $T(x) = Ax$ must have $\mathbf{R}^m$ as its range.

TRUE **FALSE**

c) If $\{a, b, c\}$ is a basis of a subspace V, then $\{a, a+b, b+c\}$ is a basis of V as well.

TRUE **FALSE**

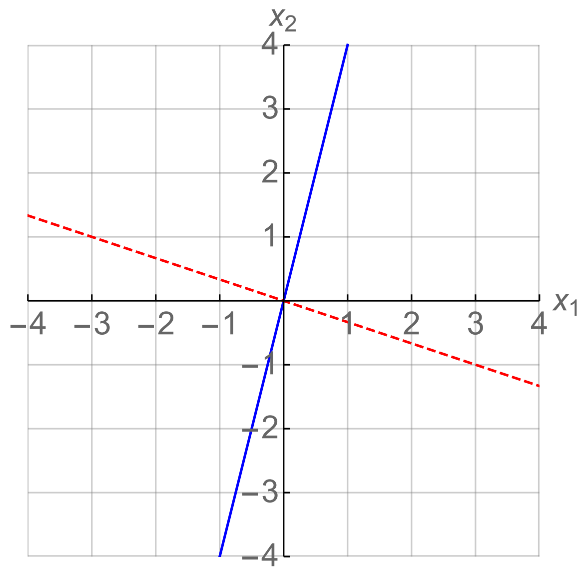
Solution.

a) True. Recall that when we say a matrix has two pivots, we mean that its RREF has two pivots. $\mathrm{rank}(A)$ is the same as number of pivots in A. $\dim(\mathrm{Nul} A)$ is the same as the number of free variables. Moreover by the Rank Theorem, $\mathrm{rank}(A) + \dim(\mathrm{Nul} A) = 10$, so $\dim(\mathrm{Nul} A) = 10 - 2 = 8$.

b) False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for $Ax = 0$, but its column space is a 2-dimensional subspace of $\mathbf{R}^3$.

c) True. Because a and b are independent, $a + b$ and a are linearly independent, and furthermore a and b are in $\mathrm{Span}\{a, a+b\}$. Next, c is independent from $\{a, b\}$, so $b + c$ is independent from $\{a, a+b\}$, meaning that $\{a, a+b, b+c\}$ is independent by the increasing span criterion. Since $a, a+b, b+c$ are all clearly in $\mathrm{Span}\{a, b, c\}$, by the basis theorem $\{a, a+b, b+c\}$ also form a span for $\mathrm{Span}\{a, b, c\} = V$. Alternatively, we could notice that a, b, and c are $\mathrm{Span}\{a, a+b, b+c\}$, and since $V = \mathrm{Span}\{a, b, c\}$ it is a three-dimensional space spanned by the set of three elements $\{a, a+b, b+c\}$, those three elements must form a basis, by the basis theorem.

3. Write a matrix A so that Col(A) is the solid blue line and Nul(A) is the dotted red line drawn below.



Solution.

We'd like to design an A with the prescribed column space Span$\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$ and null space Span$\left\{\begin{pmatrix}3\\-1\end{pmatrix}\right\}$.

We start with analyzing the null space. We can write parametric form of the null space:

$$\begin{pmatrix}x_1\\x_2\end{pmatrix} = t\begin{pmatrix}3\\-1\end{pmatrix} \quad \text{is the same as} \quad \begin{pmatrix}x_1\\x_2\end{pmatrix} = \begin{pmatrix}-3x_2\\x_2\end{pmatrix}$$

Then this implies the RREF of A must be $\begin{pmatrix}1 & 3\\0 & 0\end{pmatrix}$.

Now we need to combine the information that column space is Span$\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$. That means the second row must be 4 multiple of the first row. Therefore the second row must be $\begin{pmatrix}4 & 12\end{pmatrix}$. We conclude,

$$A = \begin{pmatrix}1 & 3\\4 & 12\end{pmatrix}$$

Note any nonzero scalar multiple of the above matrix is also a solution.

4. Let $A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix}$, and let T be the matrix transformation associated to A, so $T(x) = Ax$.

a) What is the domain of T? What is the codomain of T? Give an example of a vector in the range of T.

b) This is extra practice in case the studio finishes the rest of the worksheet early. The RREF of A is $\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

(i) Write bases for Col(A) and Nul(A).

(ii) Is there a vector in the codomain of T which is not in the range of T? Justify your answer.

Solution.

a) The domain is $\mathbf{R}^4$; the codomain is $\mathbf{R}^3$. The vector $0 = T(0)$ is contained in the range, as is

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

b) (i) First, recall that the columns of A, which correspond to pivots in the RREF of A, form a basis for Col(A). We note that the first and second column of the RREF of A contain pivots. Therefore, the first and second columns of A form a basis for Col(A). That is,

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -5 \\ 3 \\ 1 \end{pmatrix} \right\}.$$

Notice that the columns in the RREF of A do not form such basis themselves and in order to write a basis for Col(A), we need to use the corresponding columns in the matrix A itself.

In order to write a basis for Nul(A), we need to find the solution to the matrix equation $A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0$ in parametric vector form. Since x_3 and x_4 are free variables in this example. The parametric solution would be

$$\begin{cases} x_1 = -3x_3 - x_4 \\ x_2 = -x_3 - x_4 \end{cases},$$

which in vector form can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul(A) is given by

$$\left\{ \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(ii) Yes. The range of T is the column span of A, and A only has two pivots, so its column span is a 2-dimensional subspace of $\mathbf{R}^3$. Since $\dim(\mathbf{R}^3) = 3$, the range is not equal to $\mathbf{R}^3$.