MATH 1553, PRACTICE FINAL EXAM B SOLUTIONS
SPRING 2024

Name		**GT ID**	

Circle your instructor and lecture below.

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Simplify your answers as much as possible. For example, you may lose points if you do not simplify $\frac{8}{2}$ to 4, or if you do not simplify $\frac{0.1}{0.9}$ to $\frac{1}{9}$, etc.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- We use $e_1, e_2, \ldots, e_n$ to denote the standard unit coordinate vectors of $\mathbf{R}^n$.
- We will hand out loose scrap paper, but it **will not be graded**. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:00 PM on Tuesday, April 30..

____

Problem 1.

[1 point each]

TRUE or FALSE. Circle **T** if the statement is *always* true. Otherwise, answer **F**. You do not need to show work or justify your answer.

a) **T** **F** If $T : \mathbf{R}^3 \to \mathbf{R}^3$ is a linear transformation that satisfies

$$T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\2\\0\end{pmatrix}, \qquad T\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\-1\end{pmatrix}, \qquad T\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}3\\0\\0\end{pmatrix},$$

then T is one to one.

b) **T** **F** If the system $Ax = \begin{pmatrix}1\\-2\end{pmatrix}$ has a unique solution $\begin{pmatrix}3\\1\end{pmatrix}$, then the homogeneous equation $Ax = 0$ has only the trivial solution.

c) **T** **F** If A and B are $n \times n$ matrices then $\det(A+B) = \det(A) + \det(B)$.

d) **T** **F** If A is a 5×7 and $\dim(\text{Nul } A) = 4$, then $\dim(\text{Row } A) = 3$.

e) **T** **F** The set $W = \left\{ \begin{pmatrix}x\\y\\z\\w\end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - y = z - w \right\}$ is a 3-dimensional subspace of $\mathbf{R}^4$.

f) **T** **F** If A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(2-\lambda)(3-\lambda),$$

then A is diagonalizable.

g) **T** **F** Suppose W is a subspace of $\mathbf{R}^n$ and B is the matrix for orthogonal projection onto W. Then for every x in $\mathbf{R}^n$, we have $Bx = x$ or $Bx = 0$.

h) **T** **F** Each inconsistent system $Ax = b$ has exactly one least squares solution.

i) **T** **F** Any $n \times n$ matrix with n linearly independent eigenvectors in $\mathbf{R}^n$ is diagonalizable.

j) **T** **F** The vector $\begin{pmatrix}0.6\\0.4\end{pmatrix}$ is the steady state vector of the matrix $\begin{pmatrix}0.4 & 0.6\\0.6 & 0.4\end{pmatrix}$.

Solution to problem 1.

a) True. We can see this in many ways: the matrix $A = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$ has linearly independent columns / a pivot in each column / nonzero determinant / columns that clearly span $\mathbf{R}^3$.

b) True. The solution set to any consistent system $Ax = b$ is a translation of the solution set to $Ax = 0$. Since the $Ax = b$ solution set is unique, so is the solution set to $Ax = 0$.

c) False. For example,

$$\det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \qquad \text{but} \qquad \det\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 0 + 0 = 0.$$

d) True. Since $\dim(\mathrm{Nul}(A)) = 4$ and $\mathrm{Nul}A$ is a subspace of $\mathbf{R}^7$, it follows that

$$\dim((\mathrm{Nul}\ A)^\perp) = 3.$$

Since $(\mathrm{Nul}\ A)^\perp = \mathrm{Row}\ A$ this means $\dim(\mathrm{Row}\ A) = 3$.

e) True. Note $W = \mathrm{Nul}\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix}$ so it is a subspace of $\mathbf{R}^4$, and the system $\left(\begin{array}{cccc|c} 1 & -1 & -1 & 1 & 0 \end{array}\right)$ has three free variables so $\dim(W) = 3$.

f) True: A is a 3×3 matrix with 3 distinct real eigenvalues, so A is diagonalizable.

g) False: For example, if W is projection onto the x-axis in $\mathbf{R}^3$ then $B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$,

and $B\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is neither $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ nor $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$, since

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

h) False: the condition for uniqueness is that A have linearly independent columns, not that the system be inconsistent. For example, if $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ then $Ax = b$ is inconsistent but there are infinitely many least-squares solutions: $\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 3 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right)$ which has infinitely many solutions.

i) True, the n linearly independent vectors in $\mathbf{R}^n$ must form a basis for $\mathbf{R}^n$.

j) False. The steady-state vector of the matrix is $\begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$.

Problem 2.

Short answer. You do not need to show your work on (a) or (b), but briefly show your work on part (c).

a) Let $\{v_1, v_2, \ldots, v_n\}$ be vectors in $\mathbf{R}^m$. Which of the following conditions imply that these vectors are linearly independent? Circle all that apply.

$\boxed{\text{(i) The vector equation } x_1v_1 + x_2v_2 + \cdots + x_nv_n = 0 \text{ has a unique solution.}}$

$\boxed{\text{(ii) The subspace Span}\{v_1, v_2, \ldots, v_n\} \text{ has dimension } n}$.

$\boxed{\text{(iii) The RREF of the matrix } A = \begin{pmatrix} | & | & \cdots & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & \cdots & | \end{pmatrix} \text{ has a pivot in every column.}}$

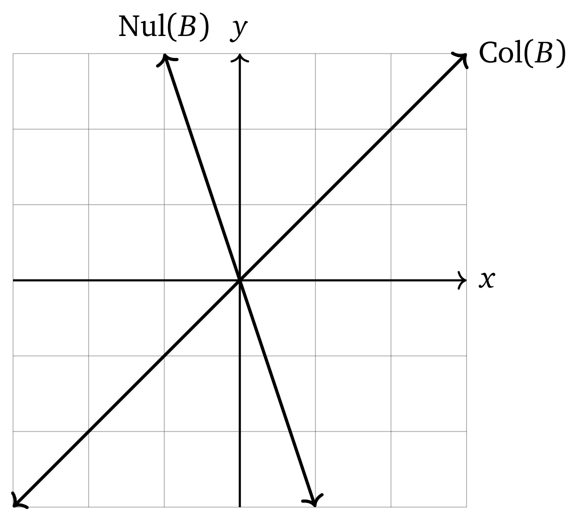
b) Let A be a 3×4 matrix. Which of the following statements *can* be true? Circle all that apply.

(i) The transformation $T : \mathbf{R}^4 \to \mathbf{R}^3$ defined by $T(x) = Ax$ is one to one.

(ii) The rank of A is equal to 2 and Nul(A) is the x-axis.

$\boxed{\text{(iii) The column space of } A \text{ and the null space of } A \text{ have the same dimension.}}$

c) The null space and column space of another matrix B are given in the picture. Write such a matrix B.



We need $\mathrm{Nul}(B) = \mathrm{Span}\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\}$ and $\mathrm{Col}(B) = \mathrm{Span}\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\}$. Many answers possible, for example $B = \begin{pmatrix} 3 & 1 \\ 3 & 1 \end{pmatrix}$

Problem 3.

Short answer. You do not need to show your work.

a) Suppose A is an $m \times n$ matrix and the only solution to the homogeneous equation $Ax = 0$ is the trivial solution $x = 0$. Let T be the matrix transformation $T(x) = Ax$. Which of the following *must* be true? Circle all that apply.

(i) T is onto.

$\boxed{\text{(ii) } T \text{ is one-to-one.}}$

$\boxed{\text{(iii) If } m = n \text{, then } A \text{ is invertible.}}$

$\boxed{\text{(iv) If } m > n \text{, then the equation } Ax = b \text{ is inconsistent for at least one } b \text{ in } \mathbf{R}^m.}$

b) The equation $\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} x = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ has least-squares solution $\widehat{x} = \begin{pmatrix} 2/5 \\ -2 \end{pmatrix}$. What is the closest vector to $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ in Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} \right\}$? Enter your answer here: $\boxed{\begin{pmatrix} 2/5 \\ 2 \\ 6/5 \end{pmatrix}}$.

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} \widehat{x} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 2/5 \\ -2 \end{pmatrix} = \begin{pmatrix} 2/5 \\ 2 \\ 6/5 \end{pmatrix}$$

c) Let W be the set of all vectors in $\mathbf{R}^3$ of the form (a, b, a) where a and b are real numbers. Which of the following is $W^\perp$?

(i) Span $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

(ii) Span $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

(iii) Nul$\begin{pmatrix} a & b & a \end{pmatrix}$

(iv) Nul$\begin{pmatrix} -1 & 0 & 1 \end{pmatrix}$

(v) none of these

The answer is (i). We see $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$, so $W^\perp = \text{Nul}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$.

Thus $x_1 = -x_3$, $x_2 = 0$, and x_3 is free, so a basis for $W^\perp$ is $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Problem 4.

Short answer. Assume that the entries in all matrices are real numbers. You do not need to show your work or justify your answers.

a) Suppose A is a 2×2 matrix with eigenvalues $\lambda = 1$ and $\lambda = 3$. What is the characteristic polynomial of A^2? A has eigenvalues 1 and 3, so A^2 has eigenvalues $1^2 = 1$ and $3^2 = 9$. There is only one possibility for its characteristic polynomial since it is 2×2 with 2 distinct eigenvalues:

$$\det(A^2 - \lambda I) = (1-\lambda)(9-\lambda).$$

b) Give an example of a 3×3 matrix with eigenvector $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

Many examples possible, for example $A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$.

c) Give an example of a 2×2 matrix that has no real eigenvalues.

Many examples possible. Any rotation matrix aside of I and $-I$ will work, for example $A = \dfrac{1}{\sqrt{2}}\begin{pmatrix}1&-1\\1&1\end{pmatrix}$ or $A = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$.

d) Give an example of a 3×3 matrix A with exactly one eigenvalue $\lambda = 2$, so that the 2-eigenspace of A is a line.

We need only one eigenvalue $\lambda = 2$, and we need $A - 2I$ to have two pivots so that we will only have one free variable in the homogeneous equation $(A-2I)x = 0$. Some examples are

$$A = \begin{pmatrix}2&1&1\\0&2&1\\0&0&2\end{pmatrix}, \qquad A = \begin{pmatrix}2&1&0\\0&2&1\\0&0&2\end{pmatrix}, \qquad A = \begin{pmatrix}2&0&0\\1&2&0\\1&1&2\end{pmatrix}.$$

Some NON-examples are $\begin{pmatrix}2&0&1\\0&2&1\\0&0&2\end{pmatrix}$ and $\begin{pmatrix}2&1&0\\0&2&0\\0&0&2\end{pmatrix}$, which have the wrong dimension for the 2-eigenspace.

Problem 5.

You do not need to show your work, and there is no partial credit except in part (d).

a) Let $T : \mathbf{R}^3 \to \mathbf{R}^3$ be a linear transformation and suppose $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$ are in the range of T. Write another nonzero vector in the range of T here: $\begin{pmatrix} \quad \\ \quad \\ \quad \end{pmatrix}$

b) Suppose that A is a 12×9 matrix and the solution set to $Ax = 0$ has dimension 7.

(i) Fill in the blank: the dimension of the column space of A is ________.

(ii) Fill in the blank: the dimension of the row space of A is ________.

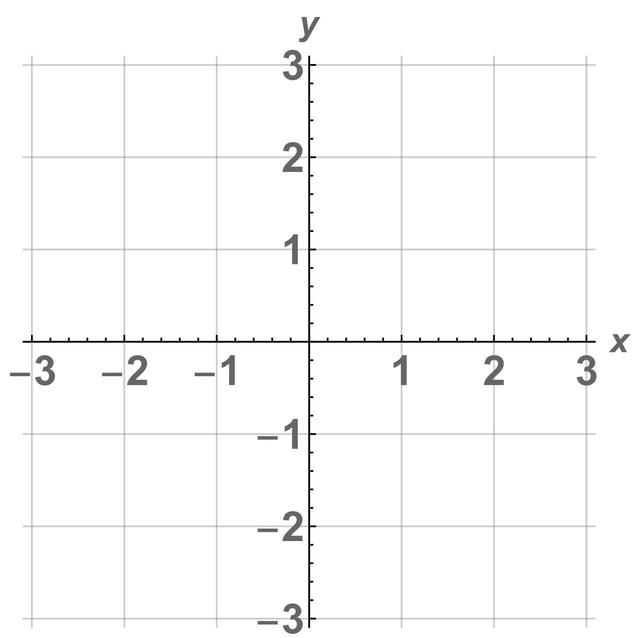
c) Suppose A is a stochastic matrix. Which of the following must be true? Circle all that apply.

(i) The sum of entries in each row of A is equal to 1.

(ii) The sum of entries in each column of A is equal to 1.

(iii) No entry of A is greater than 1.

d) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of reflection across the line $y = 3x$, and let A be the standard matrix for T. Draw each eigenspace of A precisely, and clearly label each eigenspace with its eigenvalue.



Solution:

e) Any vector in $\mathbf{R}^3$ that has 0 as its third entry is in the range of T. For example, $\begin{pmatrix} 10 \\ 1 \\ 0 \end{pmatrix}$.

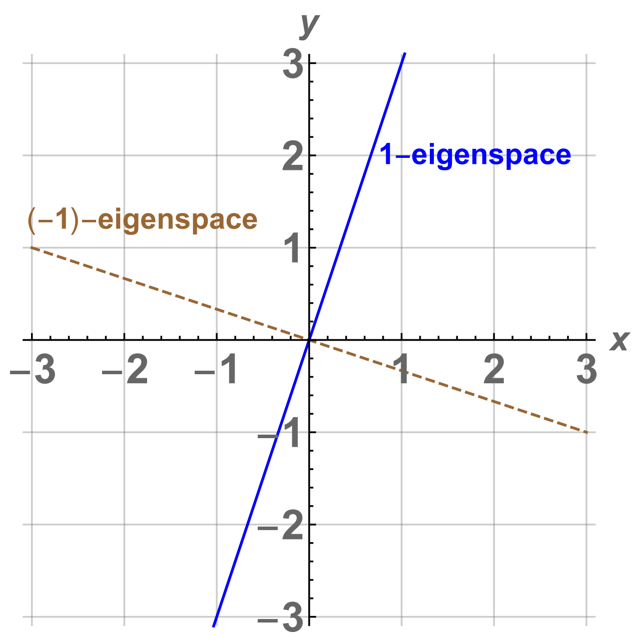
f) By the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 9.$$

We've been given $\dim(\text{Nul } A) = 7$, so
(i) $\dim(\text{Col } A) = 2$
(ii) $\dim(\text{Row } A) = \dim((\text{Nul} A)^{\perp}) = 9 - 7 = 2$.

g) (ii) must be true: it is part of the definition of stochastic matrix.
(iii) is also true since each column must sum to 1 and all entries must be 0 or greater (thus no entry can be larger than 1).

h) The reflection fixes the line $y = 3x$, so the 1-eigenspace is the line $y = 3x$. The reflection geometrically flips the line perpendicular to $y = 3x$, so the other eigenvalue is $\lambda = -1$, and the (-1)-eigenspace is the (perpendicular) line $y = -x/3$.



Problem 6.

Short answer. On parts (a), (b), and (c), you do not need to show your work and there is no partial credit. Briefly show your work in part (d).

a) Suppose that v and w are eigenvectors of a matrix A corresponding to the eigenvalues 4 and -1, respectively. Find $A(2v+3w)$ in terms of v and w.

$$A(2v+3w) = A(2v) + A(3w) = 2Av + 3Aw = 8v - 3w.$$

b) Suppose that for two 5×5 matrices A and B, we have

$$\det(A) = 3, \quad \det(A^{-1}B) = 7.$$

Find $\det(B)$.

$$\det(B) = \det(AA^{-1}B) = \det(A)\det(A^{-1}B) = 3 \cdot 7 = 21.$$

c) Suppose that A is a positive stochastic matrix with steady-state vector $\begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 350 \\ 50 \end{pmatrix}$ approach as n becomes very large?

The sum of entries of $\begin{pmatrix} 350 \\ 50 \end{pmatrix}$ is 400, so our answer is

$$400 \begin{pmatrix} 3/10 \\ 7/10 \end{pmatrix} = \begin{pmatrix} 400 \cdot 3/10 \\ 400 \cdot 7/10 \end{pmatrix} = \begin{pmatrix} 120 \\ 280 \end{pmatrix}.$$

d) Compute the orthogonal projection of $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ onto $\mathrm{Span}\left\{\begin{pmatrix} 2 \\ -3 \end{pmatrix}\right\}$.

For $u = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, the matrix for projection onto $\mathrm{Span}\{u\}$ is

$$\frac{1}{u \cdot u} uu^T = \frac{1}{4+9} \begin{pmatrix} 2 \\ -3 \end{pmatrix} \begin{pmatrix} 2 & -3 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix},$$

so

$$\mathrm{proj}_{\mathrm{Span}\{u\}} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} -16 \\ 24 \end{pmatrix} = \begin{pmatrix} -16/13 \\ 24/13 \end{pmatrix}.$$

Problem 7.

a) Find the matrix of the linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates by $90°$ counterclockwise.

b) Find the matrix of the transformation defined by $U\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x+y \\ 3y-x \\ x-3y \end{pmatrix}$.

c) Circle which transformation makes sense: $T \circ U \quad U \circ T$
Find the standard matrix for the transformation you circled, and enter it below.

d) Find A^{-1} if $A = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$.

Solution.

a) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

b) $B = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 1 & -3 \end{pmatrix}$.

c) $U \circ T$ makes sense, and $BA = \begin{pmatrix} 2 & 1 \\ -1 & 3 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 3 & 1 \\ -3 & -1 \end{pmatrix}$

d) $A^{-1} = \dfrac{1}{2(1)-(-1)(1)} \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \dfrac{1}{3}\begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{pmatrix}$.

Problem 8.

Consider the matrix

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$$

a) Find the characteristic polynomial and the eigenvalues of A.

b) For each eigenvalue of A, find one corresponding eigenvector.

c) Find an invertible 3×3 matrix C and a diagonal matrix D so that $A = CDC^{-1}$.

Solution.

a) The characteristic polynomial is

$$\det(A-\lambda I) = \det\begin{pmatrix} 1-\lambda & 0 & 0 \\ 2 & 4-\lambda & 4 \\ 1 & 1 & 1-\lambda \end{pmatrix} = (1-\lambda)\big((4-\lambda)(1-\lambda)-4\big) = (1-\lambda)\big(\lambda^2-5\lambda\big)$$
$$= (1-\lambda)(\lambda)(\lambda-5)$$

so the eigenvalues of A are $\lambda = 0$, $\lambda = 1$, and $\lambda = 5$.

b)

$$\big(A \mid 0\big) = \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 2 & 4 & 4 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ so a 0-eigenvector is } \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}.$$

$$\big(A-I \mid 0\big) = \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 2 & 3 & 4 & 0 \\ 1 & 1 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & -4 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ so a 1-eigenvector is } \begin{pmatrix} 4 \\ -4 \\ 1 \end{pmatrix}.$$

$$\big(A-5I \mid 0\big) = \left(\begin{array}{ccc|c} -4 & 0 & 0 & 0 \\ 2 & -1 & 4 & 0 \\ 1 & 1 & -4 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \text{ so a 5-eigenvector is } \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}.$$

c) $A = CDC^{-1}$ where $C = \begin{pmatrix} 0 & 4 & 0 \\ -1 & -4 & 4 \\ 1 & 1 & 1 \end{pmatrix}$ and $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 5 \end{pmatrix}$. Of course, there are other possibilities, since the order of eigenvectors in C doesn't matter as long as it is matched by the corresponding eigenvalues in D.

Problem 9.

Let $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 + x_2 + x_3 = 0 \right\}$ and $x = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

(i) Find a basis for W.

$W = \mathrm{Nul}\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, so $x_1 = -x_2 - x_3$ while x_2 and x_3 are free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{Basis} : \left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

(ii) Find x_W, the orthogonal projection of x onto W.

We solve $A^T A v = A^T x$ where $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.

$$A^T A = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}, \qquad A^T x = \begin{pmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

$$\left(A^T A \mid A^T x \right) = \left(\begin{array}{cc|c} 2 & 1 & -1 \\ 1 & 2 & -1 \end{array}\right) \xrightarrow{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 2 & 1 & -1 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & 2 & -1 \\ 0 & -3 & 1 \end{array}\right) \xrightarrow[\text{then } R_2 = -R_2/3]{R_1 = R_1 + 2R_2/3} \left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \end{array}\right), \quad v = \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix}.$$

Thus

$$x_W = Av = \begin{pmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix}.$$

(iii) Find $x_{W^\perp}$

$$x_{W^\perp} = x - x_W = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 2/3 \\ -1/3 \\ -1/3 \end{pmatrix} = \begin{pmatrix} 4/3 \\ 4/3 \\ 4/3 \end{pmatrix}.$$

Problem 10.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,0), \qquad (1,2), \qquad (3,-1).$$

Enter your answer below:

$$y = \underline{\hspace{2em}}\,x \; + \; \underline{\hspace{2em}}.$$

You must show appropriate work. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

No line goes through all three points. The corresponding (inconsistent) system is

$$\begin{aligned} 0 &= M(0) + B \\ 2 &= M(1) + B \\ -1 &= M(3) + B \end{aligned}$$

and the corresponding matrix equation is $Ax = b$ where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix}$.

We solve $A^T A\widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 10 & 4 \\ 4 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

$$\left(A^T A \mid A^T b\right) = \left(\begin{array}{cc|c} 10 & 4 & -1 \\ 4 & 3 & 1 \end{array}\right) \xrightarrow{R_1 = R_1/10} \left(\begin{array}{cc|c} 1 & 4/10 & -1/10 \\ 4 & 3 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 4R_1} \left(\begin{array}{cc|c} 1 & 4/10 & -1/10 \\ 0 & 14/10 & 14/10 \end{array}\right)$$

$$\xrightarrow{R_2 = 10R_2/14} \left(\begin{array}{cc|c} 1 & 4/10 & -1/10 \\ 0 & 1 & 1 \end{array}\right) \xrightarrow{R_1 = R_1 - 4R_2/10} \left(\begin{array}{cc|c} 1 & 0 & -5/10 \\ 0 & 1 & 1 \end{array}\right).$$

Thus $\widehat{x} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$. The line is

$$y = \frac{-x}{2} + 1.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.