MATH 1553, SPRING 2024

SAMPLE MIDTERM 2A: COVERS SECTIONS 2.5 - 3.6

Name		GT ID	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

This is a practice exam. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

The exam is not designed to test material from the previous midterm on its own. However, knowledge of the material prior to section §2.5 is necessary for everything we do for the rest of the semester, so it is fair game for the exam as it applies to §§2.5 through 3.6.

Problem 1.

For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) If A is a 30×20 matrix and $\dim(\text{Col } A) = 10$, then the null space of A is a 10-dimensional subspace of $\mathbf{R}^{20}$.

TRUE FALSE

b) If A is an $m \times n$ matrix and $m > n$, then the matrix transformation $T(x) = Ax$ cannot be one-to-one.

TRUE FALSE

c) Suppose A is a 3×2 matrix whose columns are linearly independent, and let T be the matrix transformation $T(x) = Ax$. Then

$$\left\{ T\begin{pmatrix}1\\0\end{pmatrix},\ T\begin{pmatrix}0\\1\end{pmatrix} \right\}$$

is a basis for the range of T.

TRUE FALSE

d) Suppose $T : \mathbf{R}^4 \to \mathbf{R}^2$ and $U : \mathbf{R}^2 \to \mathbf{R}^3$ are matrix transformations, and let A be the standard matrix for $U \circ T$, so $(U \circ T)(x) = Ax$. Then A is a 4×3 matrix.

TRUE FALSE

e) If A is a 3×3 matrix and $A\begin{pmatrix}1\\0\\-1\end{pmatrix} = A\begin{pmatrix}7\\-1\\2\end{pmatrix}$, then A cannot be invertible.

TRUE FALSE

Problem 2.

Parts (a), (b), and (c) are unrelated. There is no work required and no partial credit on this page.

a) (3 points) Consider the set $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x - y \geq 0 \right\}$.

(i) Does V contain the zero vector? YES NO

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? YES NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? YES NO

b) (3 points) Suppose $\{v_1, v_2, v_3\}$ is a set of vectors in $\mathbf{R}^n$. Which of the following statements are true? Clearly circle all that apply.

(i) If $\{v_1, v_2, v_3\}$ is a basis for $\mathbf{R}^n$, then $n = 3$.

(ii) If the vector equation $x_1v_1 + x_2v_2 + x_3v_3 = 0$ has the trivial solution, then $\{v_1, v_2, v_3\}$ must be linearly independent.

(iii) If $\{v_1, v_2, v_3\}$ is linearly dependent, then there is a nonzero number x_1, a nonzero number x_2, and a nonzero number x_3 so that $x_1v_1 + x_2v_2 + x_3v_3 = 0$.

c) (4 points) Suppose A is an 11×5 matrix and T is the corresponding linear transformation given by the formula $T(x) = Ax$. Which of the following statements are true? Clearly circle all that apply.

(i) $\dim(\text{Col } A) \geq \dim(\text{Nul} A)$.

(ii) If the columns of A are linearly independent, then the range of T is $\mathbf{R}^5$.

(iii) Suppose b is a vector so that the matrix equation $Ax = b$ is consistent. Then the set of solutions to $Ax = b$ must be a subspace of $\mathbf{R}^5$.

(iv) If the matrix equation $Ax = 0$ has infinitely many solutions, then $\text{rank}(A) \leq 4$.

Problem 3.

Parts (a), (b), and (c) are unrelated. You do not need to show your work.

a) (3 points total) Which of the following linear transformations are **invertible**? Circle all that apply.

(i) The transformation $T(x) = Ax$, where A is a 3×3 matrix whose columns are linearly independent.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates vectors clockwise by 10 degrees.

(iii) The matrix transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix}$.

b) (4 points) Suppose A is a 4×3 matrix and B is a 3×2 matrix, and let T be the matrix transformation $T(x) = ABx$. Which of the following must be true? Clearly circle all that apply.

(i) The column space of AB is a subspace of $\mathbf{R}^2$.

(ii) Every vector in the null space of AB is also in the null space of A.

(iii) T has domain $\mathbf{R}^2$ and codomain $\mathbf{R}^4$.

(iv) T cannot be onto.

c) (3 points) Which of the following transformations are **linear** transformations? Clearly circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x_1, x_2, x_3) = (x_1 - x_2,\ 1 - x_1,\ x_1)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^3$ given by $T(x_1, x_2) = (0,\ x_1,\ x_1)$.

(iii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ given by $T(x_1, x_2) = (x_1,\ x_1 x_2)$.

Problem 4.

You do not need to show your work on this problem. Parts (a), (b), (c), and (d) are unrelated.

a) (3 points) Suppose that A is a matrix that represents a linear transformation T from $\mathbf{R}^7$ to $\mathbf{R}^9$. In other words, T is the transformation given by the formula $T(x) = Ax$.

(i) How many rows does the matrix A have? Enter your answer here: ____________.

(ii) Suppose the reduced row echelon form of the matrix A contains 3 pivots. Apply the Rank Theorem to A to fill in the following blanks with numbers.

$$\dim(\text{Col } A) = \_\_\_\_\_\_\_\_ \qquad \dim(\text{Nul } A) = \_\_\_\_\_\_\_\_.$$

b) (2 points) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation. Which **one** of the following is a definition that T is onto?

(i) For each x in $\mathbf{R}^n$, there is a vector y in $\mathbf{R}^m$ so that $T(x) = y$.

(ii) For each x in $\mathbf{R}^n$, there is at least one vector y in $\mathbf{R}^m$ so that $T(x) = y$.

(iii) For each y in $\mathbf{R}^m$, there is at least one vector x in $\mathbf{R}^n$ so that $T(x) = y$.

c) (3 points) Let V be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form

$$\begin{pmatrix} -4x_4 \\ x_2 \\ x_2 + 6x_4 \\ x_4 \end{pmatrix}.$$

Write a basis for V.

d) (2 points) Which of the following linear transformations are one-to-one? Clearly circle all that apply.

(i) $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates vectors counterclockwise by $15°$.

(ii) $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} x.$

Problem 5.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, consider the matrix A and its reduced row echelon form given below.

$$A = \begin{pmatrix} 1 & 7 & 0 & -4 \\ -1 & -7 & 1 & 7 \\ 2 & 14 & 1 & -5 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 7 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) (2 points) Write a basis for Col A. Briefly justify your answer.

b) (4 points) Find a basis for Nul A.

c) (2 points) Write one vector x that is not the zero vector and is in the null space of A. Briefly justify your answer.

d) (2 points) Let T be the matrix transformation $T(x) = Ax$. Circle the correct answers below. You do not need to show your work on this part.

(i) The range of T is:

a point   a line   a plane   all of $\mathbf{R}^3$ all of $\mathbf{R}^4$

(ii) The range of T is a subspace of:

$\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^3$ $\mathbf{R}^4$

Problem 6.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation by 90° counterclockwise.
Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = x$.
Let $V : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation

$$V(x_1, x_2, x_3) = (x_1 - 5x_2,\ 3x_1 - 4x_3).$$

a) (2 points) Write the standard matrix A for T.
(do *not* leave your answer in terms of sine and cosine; simplify it completely)

b) (2 points) Write the standard matrix B for U.

c) (3 points) Find the standard matrix C for V.

d) (3 points) Find the standard matrix D for the transformation $W : \mathbf{R}^2 \to \mathbf{R}^2$ that first reflects vectors in $\mathbf{R}^2$ across the line $y = x$, then rotates vectors by 90° counterclockwise.

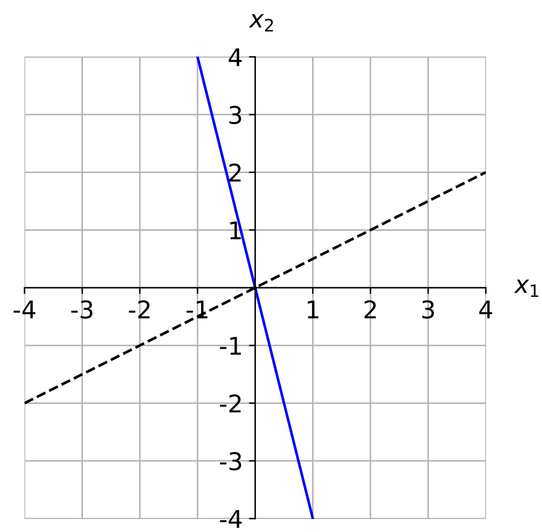
Problem 7.

Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a), (b), and (c) are unrelated.

a) (4 points) Let $A = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix}$.

Find all values of a and b so that $A^2 = A$.

b) (4 points) Write a single matrix A with the property that Col(A) is the solid line graphed below and Nul(A) is the dotted line graphed below.



c) (2 points) Give one specific example of a subspace V of $\mathbf{R}^3$ that contains the vector $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Briefly justify your answer.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.