MATH 1553, EXAM 3 SOLUTIONS
SPRING 2024

Name		**GT ID**	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, April 10.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) If A and B are $n \times n$ matrices, then

$$\det(A+B) = \det(A) + \det(B).$$

TRUE $\boxed{\text{FALSE}}$

b) Suppose A is an $n \times n$ matrix and v is an eigenvector of A. Then $-2v$ is also an eigenvector of A.

$\boxed{\text{TRUE}}$ FALSE

c) Consider the linear transformation $T(x) = Ax$ where $A = \begin{pmatrix} 3 & -1 \\ 5 & 1 \end{pmatrix}$. If S is a circle in $\mathbf{R}^2$ whose area is 4, then the area of $T(S)$ is 32.

$\boxed{\text{TRUE}}$ FALSE

d) If A is a diagonalizable $n \times n$ matrix, then every nonzero vector in $\mathbf{R}^n$ is an eigenvector of A.

TRUE $\boxed{\text{FALSE}}$

e) If A is a 3×3 matrix, then A must have exactly two complex eigenvalues.

TRUE $\boxed{\text{FALSE}}$

Solution.

a) False: for example, if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $\det(A) = \det(B) = 0$ so

$$\det(A) + \det(B) = 0, \qquad \det(A+B) = \det\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1.$$

b) True: if v is a nonzero vector in an eigenspace then $-2v$ is also a nonzero vector in the same eigenspace since eigenspaces are subspaces and therefore closed under scalar multiplication.

c) True: the area is $4|\det(A)| = 4(8) = 32$. This was basically copied from #9 of the Determinants 1 Webwork.

d) False: for example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is not an eigenvector since $A\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ which is not a scalar multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The similar-sounding true statement is that if A is diagonalizable, then it has n linearly independent eigenvectors in $\mathbf{R}^n$, however a linear combination of eigenvectors is not necessarily an eigenvector.

e) False: for example take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ which has three different real eigenvalues.

2. **Solutions are on the next page.**

a) (4 points) Suppose A is a 3×3 matrix. Which of the following statements must be true? Clearly circle all that apply.

(i) $\boxed{\text{If } \det(A) \neq 0\text{, then } A \text{ is invertible.}}$

(ii) $\boxed{\text{If } B \text{ is an } 3 \times 3 \text{ matrix, then } \det(AB) = \det(A)\det(B).}$

(iii) If A is invertible, then $\det(A^{-1}) = -\det(A)$.

(iv) $\boxed{\text{If } v \text{ and } w \text{ are eigenvectors of } A \text{ satisfying } Av = 8v \text{ and } Aw = 3w\text{, then } v + w \text{ cannot be an eige}}$

b) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Find

$$\det\begin{pmatrix} 2d-3a & 2e-3b & 2f-3c \\ a & b & c \\ g & h & i \end{pmatrix}.$$

Clearly circle your answer below.

(i) 1/2 (ii) −1/2 (iii) 2 (iv) −2 (v) 4

$\boxed{\text{(vi) } -4}$ (vii) 6 (viii) −6 (ix) −12 (x) none of these

c) (2 points) Find the value of a so that

$$a \cdot \det\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 5 & 3 & 10 \end{pmatrix} = \det\begin{pmatrix} 2 & -2 & 4 \\ 0 & 2 & 2 \\ 10 & 6 & 20 \end{pmatrix}.$$

(i) $a = 1/8$ (ii) $a = 1/4$ (iii) $a = 1/2$ (iv) $a = 1$ (v) $a = 2$

(vi) $a = 4$ $\boxed{\text{(vii) } a = 8}$ (viii) −12 (ix) none of these

d) (2 points) Find the area of the triangle with vertices $(1,1)$, $(2,3)$, and $(3,-2)$. Clearly circle your answer below.

(i) 1 (ii) 1/2 (iii) 3 (iv) 5 (v) 5/2 (vi) 7 $\boxed{\text{(vii) } 7/2}$

Solutions to Problem 2.

a) (i) is true and is a fundamental result of Chapter 4.
(ii) is true and is a fundamental property of determinants.
(iii) is not necessarily true, for example if $A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ then $\det(A) = 2$ but $\det(A^{-1}) = \det\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix} = 1/2$, so $\det(A^{-1}) \neq -\det(A)$.
(iv) is true: the sum of two eigenvectors from different eigenspaces is never an eigenvector. $Av = 8v$ and $Aw = 3w$, so

$$A(v+w) = Av + Aw = 8v + 3w.$$

Since v and w correspond to different eigenvalues they are automatically linearly independent, so it is not possible for $8v + 3w$ to be a scalar multiple of $v + w$.

b) This is nearly identical to #2b from Sample Midterm 3B and #7 from the Determinants 1 Webwork. To get from the first matrix to the final matrix, we first do a row swap, then we scale the first row by a factor of 2, followed by a row-replacement. Therefore the answer is $2(-1)(2) = -4$.

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2, \text{ so}$$

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow[\text{new det is } -2]{R_1 \leftrightarrow R_2} \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \xrightarrow[\text{new det is } -2(2) = -4]{R_1 = 2R_1} \begin{pmatrix} 2d & 2e & 2f \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$\xrightarrow[\text{det is still } -4]{R_1 = R_1 - 3R_2} \begin{pmatrix} 2d-3a & 2e-3b & 2f-3c \\ a & b & c \\ g & h & i \end{pmatrix}.$$

c) This is #6 from the Determinants 1 Webwork with slightly changed numbers (and also similar to a Quiz 6 problem). To get from $\begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 1 \\ 5 & 3 & 10 \end{pmatrix}$ to $\begin{pmatrix} 2 & -2 & 4 \\ 0 & 2 & 2 \\ 10 & 6 & 20 \end{pmatrix}$, we must multiply each of the three rows of the original matrix by 2, which is equivalent to multiplying the determinant by $2 \cdot 2 \cdot 2 = 8$. Therefore, $a = 8$.

d) This is #8 from Determinants 1 Webwork with slighly changed numbers. We form vectors from the 1st to 2nd vertex and also from the 1st to 3rd vertex, getting $v_1 = \begin{pmatrix} 2-1 \\ 3-1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 3-1 \\ -2-1 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$. The area of the triangle is half the area of the associated parallelogram:

$$\text{Area} = \frac{1}{2}\left|\det\begin{pmatrix} 1 & 2 \\ 2 & -3 \end{pmatrix}\right| = \frac{1}{2}(7) = \frac{7}{2}.$$

3. **Solutions are on the next page.**

a) (3 points) Suppose x is a **nonzero** vector in $\mathbf{R}^n$ and A is an $n \times n$ matrix. Which of the following conditions guarantee that x is an eigenvector of A? Clearly circle all that apply.

(i) $\boxed{x \text{ is in the null space of } A.}$

(ii) $\boxed{Ax \text{ is a scalar multiple of } x.}$

(iii) The set $\{x, Ax\}$ is linearly independent.

b) (3 points) Let A be an $n \times n$ matrix so that $\det(A - 2I) = 6$. Which of the following statements **must** be true? Clearly circle all that apply.

(i) $\lambda = 2$ is an eigenvalue of the matrix A.

(ii) $\boxed{\text{The only solution to } (A-2I)x = 0 \text{ is the trivial solution } x = 0.}$

(iii) The matrix A is invertible.

c) (4 points) Let C be a 3×3 matrix with eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$, and suppose that the 2-eigenspace of C is $\mathrm{Span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$.

(i) Find $C\begin{pmatrix}1\\0\\1\end{pmatrix}$. Clearly circle your answer below.

(I) $\begin{pmatrix}-1\\0\\-1\end{pmatrix}$ $\boxed{\text{(II)} \begin{pmatrix}2\\0\\2\end{pmatrix}}$ (III) $\begin{pmatrix}4\\0\\4\end{pmatrix}$ (IV) $\begin{pmatrix}2\\0\\0\end{pmatrix}$ (V) Not enough information

(ii) Must it be true that C is diagonalizable?

$\boxed{\text{YES}}$ NO NOT ENOUGH INFORMATION

(iii) Which of the following vectors **could** be in the (-1)-eigenspace of C? Clearly circle all that apply.

(I) $\begin{pmatrix}4\\0\\5\end{pmatrix}$ $\boxed{\text{(II)} \begin{pmatrix}6\\1\\1\end{pmatrix}}$ (III) $\begin{pmatrix}0\\0\\3\end{pmatrix}$ $\boxed{\text{(IV)} \begin{pmatrix}2\\-7\\5\end{pmatrix}}$.

Solutions to Problem 3.

a) (i) Yes, by the definition of eigenvector: if x is a nonzero vector in the null space of A then $Ax = 0x$, so x is an eigenvector for the eigenvalue $\lambda = 0$.

(ii) Yes, directly by the definition of eigenvector: if Ax is a scalar multiple of x then $Ax = \lambda x$ for some scalar λ (where λ real since A has real entries and x is in $\mathbf{R}^n$).

(iii) No, for example if $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $x = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ then x is not an eigenvector of A but $\{x, Ax\}$ is $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ which is linearly independent.

b) (i) This is not necessarily true. In fact, $\lambda = 2$ cannot be an eigenvalue of A because $\det(A - 2I) \neq 0$.

(ii) True: $\det(A - 2I) \neq 0$, so $A - 2I$ is invertible which means that the only solution to $(A - 2I)x = 0$ is the trivial solution.

(iii) This is not necessarily true, because we are only given information about $A - 2I$ and not given any information about $A - 0I$. For example, the matrix $A = \begin{pmatrix} 0 & 0 \\ 0 & 5 \end{pmatrix}$ is not invertible, yet it satisfies $\det(A - 2I) = \det \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix} = -6$.

c) (i) $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ is in Span$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ which is the 2-eig. of C,so $C \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 2 \end{pmatrix}$.

(ii) Yes, C must be diagonalizable. We are told that the 2-eigenspace is a plane, so $\lambda = 2$ has geometric multiplicity 2. Since $\lambda = -1$ is also an eigenvalue of the 3×3 matrix C, it gives another eigenspace and therefore the sum of geometric multiplicities is 3 and C is diagonalizable.

(iii) The vectors (II) and (IV) are possible.
Note that $\begin{pmatrix} 4 \\ 0 \\ 5 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$ are nonzero vectors in the xz-plane, so they are in the 2-eigenspace rather than the (-1)-eigenspace.

(Solution is continued on the next page)

For (II) and (IV), the vector is not in the 2-eigenspace so it **could** be in the (-1)-eigenspace. In fact, if we *really* wanted, then we could actually construct a C with 2-eigenspace equal to the xz-plane and (-1)-eigenspace equal to the span of $\begin{pmatrix} 6 \\ 1 \\ 1 \end{pmatrix}$, by using the Diagonalization Theorem:

$$C = \begin{pmatrix} 1 & 0 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 6 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -18 & 0 \\ 0 & -1 & 0 \\ 0 & -3 & 2 \end{pmatrix}.$$

Similarly, we *could* construct a matrix C with 2-eigenspace equal to the xz-plane and (-1)-eigenspace equal to the span of $\begin{pmatrix} 2 \\ -7 \\ 5 \end{pmatrix}$:

$$C = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -7 \\ 0 & 1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 0 & -7 \\ 0 & 1 & 5 \end{pmatrix}^{-1} = \frac{1}{7} \begin{pmatrix} 14 & 6 & 0 \\ 0 & -7 & 0 \\ 0 & 15 & 4 \end{pmatrix}.$$

Of course, it was not necessary at all to do these last steps, they are just here to give examples where (II) and (IV) occur in reality.

4. **Full solutions are on the next page.**

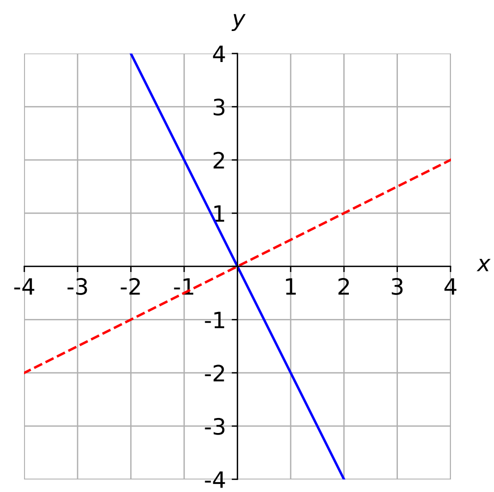
a) (4 points) Let A be the matrix that reflects vectors in $\mathbf{R}^2$ across the line $y = -2x$.

(i) Write the eigenvalues of A in the space below.

$$\boxed{1 \text{ and } -1}.$$

(ii) Draw two linearly independent eigenvectors of A on the graph below. For this problem, you only need to draw the eigenvectors (you do not need to label which eigenspace they are in).

The blue solid line is the 1-eigenspace, and the red dashed line is the (-1)-eigenspace. A correct answer must have one nonzero vector along the solid line and one nonzero vector along the dashed line.



b) (2 points) Which **one** of the following matrices A has a 3-eigenspace that is a plane? Clearly circle your answer.

(i) $A = \begin{pmatrix} 3 & 0 \\ 1 & 3 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 3 & 3 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 4 \end{pmatrix}$ $\boxed{\text{(iv) } A = \begin{pmatrix} 3 & 6 & 3 \\ 0 & 3 & 0 \\ 0 & 6 & 6 \end{pmatrix}}$

c) (4 points) Let $A = \begin{pmatrix} 0.4 & 0.1 \\ 0.6 & 0.9 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 100 \\ 40 \end{pmatrix}$ approach as n gets very large? Write your answer in the space below. Show your work on this part, and fully simplify any fractions that are in your final answer.

$$\boxed{\begin{pmatrix} 20 \\ 120 \end{pmatrix}}$$

Solutions to Problem 4.

a) This is #3b of the 4.1-5.1 Worksheet done with a different line, and its method is identical.

(i) The eigenvalues are 1 and -1. Since A implements reflection across the line $y=-2x$, it will keep fixed all vectors along the line $y=-2x$ (i.e. this line is the 1-eigenspace) and will flip all vectors along the perpendicular line through the origin, which is the line $y=x/2$ (i.e. this is the (-1)-eigenspace). Since A is a 2×2 matrix, it has a max of two different eigenvalues, so 1 and -1 are the only eigenvalues.

(ii) Any nonzero vector along the line $y=-2x$ is in the 1-eigenspace. Any nonzero vector along the line $y=x/2$ is in the (-1)-eigenspace. However, the student does not need to indicate which eigenspace each vector is in, they are only being asked to graph a vector in each eigenspace.

b) The answer is (iv). For the matrices in (i), (ii), and (iii), the augmented matrix $(A-3I \mid 0)$ gives exactly one free variable in the solution set to $(A-3I)x=0$, so the 3-eigenspace is a line.

However, in (iv), we see

$$(A-3I \mid 0)=\left(\begin{array}{ccc|c} 0 & 6 & 3 & 0\\ 0 & 0 & 0 & 0\\ 0 & 6 & 3 & 0\end{array}\right)\xrightarrow{RREF}\left(\begin{array}{ccc|c} 0 & 1 & \frac{1}{2} & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{array}\right),$$

so the solution set to $(A-3I)x=0$ has two free variables. In other words, the 3-eigenspace is a plane.

c) This is essentially #4c from Sample Midterm 3B with slightly changed numbers. It is also very similar to #4d from Sample Midterm 3A.
$(A-I \mid 0)=\left(\begin{array}{cc|c} -0.6 & 0.1 & 0\\ 0.6 & -0.1 & 0\end{array}\right)\xrightarrow{RREF}\left(\begin{array}{cc|c} 1 & -1/6 & 0\\ 0 & 0 & 0\end{array}\right)$, so the 1-eigenspace is given by $x_1=\dfrac{x_2}{6}$ where x_2 is free. Therefore, $v=\begin{pmatrix}1/6\\1\end{pmatrix}$ spans the 1-eigenspace and the steady-state vector is

$$w=\frac{1}{1/6+1}\begin{pmatrix}1/6\\1\end{pmatrix}=\frac{1}{7/6}\begin{pmatrix}1/6\\1\end{pmatrix}=\frac{6}{7}\begin{pmatrix}1/6\\1\end{pmatrix}=\begin{pmatrix}1/7\\6/7\end{pmatrix}.$$

Therefore, as n gets very large:

$$A^n\begin{pmatrix}100\\40\end{pmatrix}\to(100+40)w=140\begin{pmatrix}1/7\\6/7\end{pmatrix}=\begin{pmatrix}20\\120\end{pmatrix}.$$

For this problem, we copied #6 of Sample Midterm 3A, then changed the numbers slightly (it is also almost identical to #6 from Sample Midterm 3B).

5. Let $A = \begin{pmatrix} 2 & 0 & -6 \\ 3 & 5 & 6 \\ 0 & 0 & 5 \end{pmatrix}$.

a) Find all eigenvalues of A.
Cofactor expansion along the 3rd row gives us

$$\det(A - \lambda I) = (5-\lambda)[(2-\lambda)(5-\lambda)],$$

so the eigenvalues are $\lambda = 2$ and $\lambda = 5$.

b) Find a basis for each eigenspace of A.

$$\lambda = 2: \ \big(A - 2I \mid 0\big) = \left(\begin{array}{ccc|c} 0 & 0 & -6 & 0 \\ 3 & 3 & 6 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{ccc|c} 3 & 3 & 6 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

so $x_1 = -x_2$, x_2 is free, and $x_3 = 0$. The 2-eigenspace has basis $\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \right\}$.

$$\lambda = 5: \ \big(A - 5I \mid 0\big) = \left(\begin{array}{ccc|c} -3 & 0 & -6 & 0 \\ 3 & 0 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right),$$

so $x_1 = -2x_3$ while x_2 and x_3 are free. The 5-eigenspace has basis $\left\{ \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$.

c) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

We form C using linearly independent eigenvectors and form D using the eigenvalues written **in the corresponding order**. Many answers are possible. For example,

$$C = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

or

$$C = \begin{pmatrix} -2 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Part (a) is basically #5b from Sample Midterm 3B. Part (b) is nearly the same problem as #5b from Sample Midterm 3A and #5a from Sample Midterm 3B.

6. **a)** Find **all** the values of c such that the matrix

$$A = \begin{pmatrix} 4 & 8 & 0 \\ 2 & c & 0 \\ 1 & 6 & c \end{pmatrix}$$

satisfies $\det(A) = 20$.

Solution: We want

$$\begin{aligned} 20 &= \det(A) \\ 20 &= c(4c-16) \\ 0 &= 4c^2 - 16c - 20 \\ 0 &= c^2 - 4c - 5 \\ 0 &= (c-5)(c+1) \end{aligned}$$

Therefore, $c = 5$ or $c = -1$.

b) Find the complex eigenvalues of

$$A = \begin{pmatrix} 4 & -1 \\ 17 & 2 \end{pmatrix}.$$

Fully simplify your answer for the eigenvalues. For the eigenvalue with **negative** imaginary part, write one eigenvector v. Enter your answers below.

The eigenvalues are: $\boxed{3 \pm 4i}$. $\qquad v = \begin{pmatrix} 1 \\ 1+4i \end{pmatrix}$.

Solution:

$$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 6\lambda + 25,$$

$$\lambda = \frac{-(-6) \pm \sqrt{(-6)^2 - 25(4)(1)}}{2} = \frac{6 \pm \sqrt{36-100}}{2} = \frac{6 \pm \sqrt{-64}}{2} = \frac{6 \pm 8i}{2} = 3 \pm 4i.$$

For $\lambda = 3 - 4i$ we have

$$\left(A - (3-4i)I \mid 0\right) = \left(\begin{array}{cc|c} 1+4i & -1 & 0 \\ (*) & (*) & 0 \end{array}\right) = \left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array}\right).$$

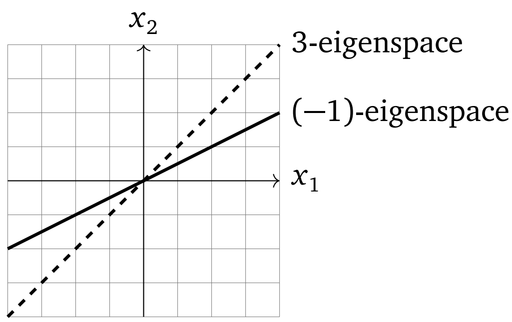
By the 2×2 eigenvector trick, $\begin{pmatrix} -b \\ a \end{pmatrix}$ is an eigenvector, so a correct answer is $\begin{pmatrix} 1 \\ 1+4i \end{pmatrix}$.

Other correct answers include

$$\begin{pmatrix} -1 \\ -1-4i \end{pmatrix}, \quad \begin{pmatrix} 1-4i \\ 17 \end{pmatrix}, \quad \begin{pmatrix} -1+4i \\ -17 \end{pmatrix}.$$

Parts (a) and (b) are nearly identical to #5a and #7b (respectively) from Sample Midterm 3A.

7. **a)** (5 points) Find the matrix A whose (-1)-eigenspace is the **solid** line below and whose 3-eigenspace is the **dashed** line below.
(Note: each square in the grid has sides of length 1)



Solution: We're given eigenvalues and eigenvectors:

$$(-1)\text{-eigenspace} = \text{Span}\left\{\begin{pmatrix}2\\1\end{pmatrix}\right\}, \qquad 3\text{-eigenspace} = \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}.$$

So our 2×2 matrix A is diagonalizable. We put the eigenvectors as columns of C and the corresponding eigenvalues as the diagonal entries of D, and compute

$$A = CDC^{-1} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & 3\end{pmatrix}\begin{pmatrix}1 & -1\\-1 & 2\end{pmatrix} = \begin{pmatrix}2 & 1\\1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\-3 & 6\end{pmatrix} = \begin{pmatrix}-5 & 8\\-4 & 7\end{pmatrix}.$$

b) (5 points) Axel and Billy are magicians who compete for customers in a group of 200 people. Today, Axel has 160 customers and Billy has 40 customers. Each day:

- 30% of Axel's customers keep attending Axel's show, while 70% of Axel's customers switch to Billy's show.
- 80% of Billy's customers attend Billy's show, while 20% of Billy's customers switch to Axel's show.

(i) Write a positive stochastic matrix B and a vector x so that Bx will give the number of customers for Axel's show and Billy's show (in that order) tomorrow. You do not need to compute Bx.

$$B = \begin{pmatrix}0.3 & 0.2\\0.7 & 0.8\end{pmatrix}, \qquad x = \begin{pmatrix}160\\40\end{pmatrix}.$$

(ii) Find the steady-state vector w for B. Write your answer in the space below.
Solution: $\left(B - I \mid 0\right) = \left(\begin{array}{cc|c}-0.7 & 0.2 & 0\\0.7 & -0.2 & 0\end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c}1 & -2/7 & 0\\0 & 0 & 0\end{array}\right)$, so for the 1-eigenspace we get $x_1 = \frac{2}{7}x_2$ where x_2 is free, thus the vector $v = \begin{pmatrix}2/7\\1\end{pmatrix}$ spans the 1-eigenspace. The steady state vector is

$$w = \frac{1}{2/7+1}\begin{pmatrix}2/7\\1\end{pmatrix} = \frac{1}{9/7}\begin{pmatrix}2/7\\1\end{pmatrix} = \begin{pmatrix}2/9\\7/9\end{pmatrix}.$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.