MATH 1553, EXAM 1 SOLUTIONS
SPRING 2024

Name		GT ID	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, February 7.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) The set of all solutions (x, y, z) to the following linear equation is a line in $\mathbf{R}^3$:

$$4x - y + z = 1.$$

TRUE $\boxed{\text{FALSE}}$

b) If a consistent system of linear equations has more variables than equations, then it must have infinitely many solutions.

$\boxed{\text{TRUE}}$ FALSE

c) Suppose v_1, v_2, and b are vectors in $\mathbf{R}^n$ with the property that b is in Span$\{v_1, v_2\}$. Then the vector $-10b$ must be a linear combination of v_1 and v_2.

$\boxed{\text{TRUE}}$ FALSE

d) If the RREF of an augmented matrix has final row $\left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \end{array}\right)$, then the corresponding system of linear equations must have infinitely many solutions.

TRUE $\boxed{\text{FALSE}}$

e) Suppose that A is a 3×3 matrix and there is a vector b in $\mathbf{R}^3$ so that the equation $Ax = b$ has exactly one solution. Then the only solution to the homogeneous equation $Ax = 0$ is the trivial solution.

$\boxed{\text{TRUE}}$ FALSE

Solution.

Two of these were slight modifications of sample exam true/false questions, and two others were slight modifications of Webwork true/false.

a) False: the augmented matrix $\left(\begin{array}{ccc|c} 4 & -1 & 1 & 1 \end{array}\right)$ has one pivot, so the solution set has 2 free variables, therefore the equation $4x - y + z = 1$ defines a **plane** in $\mathbf{R}^3$.

b) True: a **consistent** system of linear equations that has more variables than equations is guaranteed to have at least one free variable and therefore infinitely many solutions.

c) True: a span of vectors contains all linear combinations of its vectors, so if b is in Span$\{v_1, v_2\}$ then so is $-10b$. If we wanted to be mathematically precise, we could note that since $c_1v_1 + c_2v_2 = b$ for some c_1 and c_2, it follows that $-10c_1v_1 - 10c_2v_2 = -10b$ so $-10b$ is also in the span of v_1 and v_2.

d) False: for example, the system for the augmented matrix below no solutions:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

e) True: the solution set to $Ax = b$ is a translation of the solution set to $Ax = 0$ and vice versa, so if $Ax = b$ has just one solution then so does $Ax = 0$, hence the trivial solution $x = 0$ is the only solution to $Ax = 0$.

2. **Full solutions are on the next page.**

a) (4 points) Which of the following matrices are in reduced row echelon form? Clearly circle **all** that apply.

(i) $\left(\begin{array}{cccc|c} 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right)$

(ii) $\left(\begin{array}{cccc|c} 1 & -5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$

[(iii)] $\left(\begin{array}{ccc|c} 0 & 0 & 1 & -3 \end{array}\right)$

[(iv)] $\left(\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right)$

b) (2 points) Consider the vector equation in x and y given by

$$x\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}.$$

Which **one** of the following describes the solution set to the vector equation?

[(i) A point in $\mathbf{R}^2$] (ii) A line in $\mathbf{R}^2$ (iii) All of $\mathbf{R}^2$

(iv) A point in $\mathbf{R}^3$ (v) A line in $\mathbf{R}^3$ (vi) A plane in $\mathbf{R}^3$

c) (2 points) A system of linear equations in the variables x_1, x_2, x_3 has a solution set with parametric form

$$x_1 = 3 - x_3 \qquad x_2 = x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Which **one** of the following is a solution to the system of equations? Clearly circle your answer.

(i) $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ [(ii) $\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$] (iii) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (iv) $\begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}$ (v) $\begin{pmatrix} 4 \\ 2 \\ 2 \end{pmatrix}$

d) (2 points) Consider a consistent system of three linear equations in four variables. The system corresponds to an augmented matrix whose RREF has two pivots. Complete the following statements by clearly circling the **one** correct answer in each case.

(i) The solution set for the system is:

a point a line [a plane] all of $\mathbf{R}^3$ all of $\mathbf{R}^4$

(ii) Each solution to the system of linear equations is in:

$\mathbf{R}$ $\mathbf{R}^2$ $\mathbf{R}^3$ [$\mathbf{R}^4$]

Solution to Problem 2.

a) (iii) and (iv) only.
The matrices in (i) and (ii) not in RREF because, in each case, the pivot in the second row has a nonzero entry directly above it.

b) Solving the corresponding augmented system gives us

$$\left(\begin{array}{cc|c} 1 & 0 & 5 \\ -1 & 0 & -5 \\ 0 & 1 & 0 \end{array}\right) \xrightarrow{R_2=R_2+R_1} \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

Thus $x = 5$ and $y = 0$, so the only solution is the single point $(5,0)$ in $\mathbf{R}^2$.

c) Note (ii) is correct because when $x_3 = 1$ we get $x_2 = 1$ and $x_1 = 3-1 = 2$.
We see (i) is wrong because when $x_3 = 1$ we need $x_1 = 3-1 = 2 \neq 3$, so $\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$ is not a solution.
We see (iii) is wrong since when $x_3 = 0$ we need $x_1 = 3-0 = 3$.
We see (iv) is wrong immediately since $x_2 \neq x_3$.
We see (v) is wrong because when $x_3 = 2$ we need $x_1 = 3-2 = 1$.

d) There are 4 variables and the left side of the augment bar matrix has both of the 2 pivots (since the system is consistent), so there are 2 free variables.
(i) Since there are 2 free variables, the solution set is a plane.
(ii) There are 4 variables total, so each solution is in $\mathbf{R}^4$.

3. Short answer. Parts (a) through (d) are unrelated. Briefly show your work on part (a).

a) (2 pts) Find $\begin{pmatrix} 1 & 2 \\ 0 & 5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix}$. Briefly show work. Enter your answer here: $\boxed{\begin{pmatrix} 9 \\ 5 \\ -6 \end{pmatrix}}$

b) (2 pts) Let v_1, v_2 and b be nonzero vectors in $\mathbf{R}^n$. Suppose that v_1 and v_2 are **not** scalar multiples of each other and that $c_1v_1 + c_2v_2 = b$ for some real numbers c_1, c_2. Answer the following questions by circling the **one** correct answer each time.

(i) If $\mathrm{Span}\{v_1, v_2\} = \mathrm{Span}\{v_1, b\}$, then

$c_1 = 0$ $\qquad c_1 \neq 0$ $\qquad \boxed{c_2 \neq 0}$ $\qquad c_2 = 0.$

(ii) If $\mathrm{Span}\{v_2, b\}$ does **not** contain v_1, then

$\boxed{c_1 = 0}$ $\qquad c_1 \neq 0$ $\qquad c_2 \neq 0$ $\qquad c_2 = 0.$

c) (2 pts) In the spaces below, write 3 **different** vectors v_1, v_2, v_3 in $\mathbf{R}^3$ with the property that $\mathrm{Span}\{v_1, v_2, v_3\}$ is a line.
Many possibilities, for example

$$v_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}.$$

d) (4 pts) Consider the matrices $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 100 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 0 \end{pmatrix}$.

Determine whether the following statements are true or false, and clearly circle the appropriate answer.

(i) The span of the columns of A is $\mathbf{R}^3$.

$\boxed{\text{TRUE}}$ $\qquad$ FALSE

(ii) The span of the columns of B is $\mathbf{R}^2$.

TRUE $\qquad$ $\boxed{\text{FALSE}}$

(iii) For the vector $b = \begin{pmatrix} 2 \\ 7 \\ 2024 \end{pmatrix}$, the equation $Ax = b$ has exactly one solution.

$\boxed{\text{TRUE}}$ $\qquad$ FALSE

(iv) There is a vector d in $\mathbf{R}^3$ such that the matrix equation $Bx = d$ is inconsistent.

$\boxed{\text{TRUE}}$ $\qquad$ FALSE

Solution to Problem 3.

a) $\begin{pmatrix} 1 & 2 \\ 0 & 5 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 7 \\ 1 \end{pmatrix} = 7\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix} + \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 9 \\ 5 \\ -6 \end{pmatrix}$.

b) Since v_1 and v_2 are nonzero vectors and neither is a scalar multiple of the other, we know $\text{Span}\{v_1, v_2\}$ is a plane.
(i) If $\text{Span}\{v_1, v_2\} = \text{Span}\{v_1, b\}$, then $\text{Span}\{v_1, b\}$ is a plane (from above), so b cannot be a scalar multiple of v_1. This means that $c_2 \neq 0$, because if $c_2 = 0$ then $c_1v_1 + 0 = b$, which would mean b is a scalar multiple of v_1.
(ii) From the fact $c_1v_1 + c_2v_2 = b$ we get $\boxed{c_1v_1 = b - c_2v_2}$. Since $\text{Span}\{v_2, b\}$ does not contain v_1, we must have $c_1 = 0$. Otherwise, v_1 would be a linear combination of v_2 and b because the boxed equation would give

$$v_1 = \frac{1}{c_1}b - \frac{c_2}{c_1}v_2.$$

c) This is nearly verbatim #2a from the 2.1-2.2 Worksheet. We can just take any three different scalar multiples of the same vector. For example,

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad v_2 = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \qquad v_3 = \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix}.$$

d) (i) True, since A has a pivot in each of its three rows.
(ii) False: the span of the columns of B is a plane in $\mathbf{R}^3$, which is **not** the same as $\mathbf{R}^2$. Every vector in $\mathbf{R}^2$ has two entries, while every vector in the column span of B has three entries. This was nearly copied from slide 12 of the section 1.1 PDF.
(iii) True: $(A|b)$ has a pivot in every column except the rightmost column, so $Ax = b$ has exactly one solution.
(iv) True: for example if $d = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ then $(B|d) = \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$.

4. **a)** (2 points) Find all values of c so that $\begin{pmatrix}2\\5\end{pmatrix}$ is in Span$\left\{\begin{pmatrix}1\\-1\end{pmatrix}, \begin{pmatrix}3\\c\end{pmatrix}\right\}$. Clearly circle the **one** correct answer below.

(i) $c = 0$ only (ii) $c = -3$ only (iii) $c = 1$ only (iv) $c = 3$ only

(v) All c except 0 (vi) All c except -3 (vii) All c except 3 (viii) All real c

Solution: We solve the system with augmented matrix below:

$$\left(\begin{array}{cc|c} 1 & 3 & 2 \\ -1 & c & 5 \end{array}\right) \xrightarrow{R_2 = R_2 + R_1} \left(\begin{array}{cc|c} 1 & 3 & 2 \\ 0 & c+3 & 7 \end{array}\right)$$

This is consistent if and only if $c + 3 \neq 0$, so $c \neq -3$.

b) (4 points) Write an augmented matrix in RREF so that the solution set to the corresponding system of linear equations has parametric form

$$x_1 = 2x_2 - 1, \qquad x_2 = x_2 \quad (x_2 \text{ real}).$$

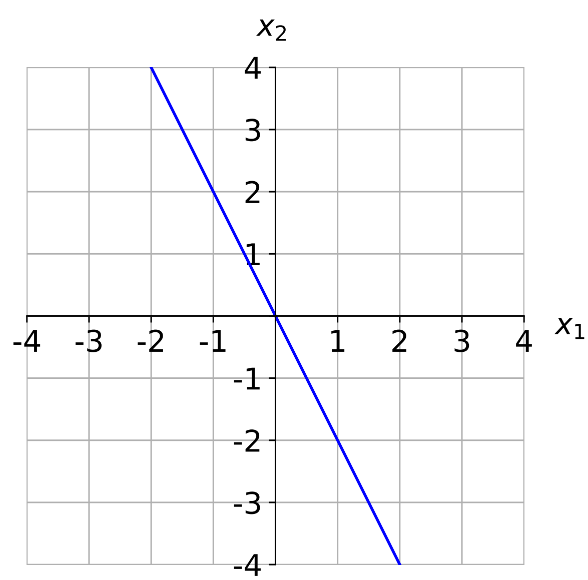
Solution: Multiple examples possible, for example

$$\left(\begin{array}{cc|c} 1 & -2 & -1 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{cc|c} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right).$$

c) (4 points) Suppose A is a 2×2 matrix whose RREF has one pivot, and suppose that $x = \begin{pmatrix}1\\-2\end{pmatrix}$ is one solution to $Ax = 0$.

On the graph below, very carefully draw the set of all solutions to $Ax = 0$.

Solution: A has two columns but one pivot, so $Ax = 0$ has one variable in its solution set. Therefore, the solution set is the line through the origin and $(1, -2)$.



The rest of the exam is free response. Unless told otherwise, show your work! A correct answer without appropriate work will receive little or no credit, even it is correct.

5. (10 points) Consider the system of linear equations in x and y given by

$$x - 3y = h$$
$$4x + ky = 20,$$

where h and k are real numbers.

This problem was copied and pasted from #5 in Sample Midterm 1A, with slightly changed numbers.

Before doing any of the three parts, we do one step of row-reduction.

$$\left(\begin{array}{cc|c} 1 & -3 & h \\ 4 & k & 20 \end{array}\right) \xrightarrow{R_2 = R_2 - 4R_1} \left(\begin{array}{cc|c} 1 & -3 & h \\ 0 & k+12 & 20-4h \end{array}\right).$$

a) Find all values of h and k (if there are any) so that the system is inconsistent.

The system is inconsistent when the rightmost column has a pivot, thus $k+12 = 0$ and $20 - 4h \neq 0$.

$$\boxed{h \neq 5, \qquad k = -12}.$$

b) Find all values of h and k (if there are any) so that the system has infinitely many solutions.

The system has infinitely many solutions when the rightmost column does not have a pivot and some other column (in this case, the second) does not either. This means the second row is all zeros, so $k + 12 = 0$ and $20 - 4h = 0$.

$$\boxed{h = 5, \qquad k = -12}.$$

c) Find all values of h and k (if there are any) so that the system has exactly one solution.

The system has a unique solution when the system has two pivots and they are both to the left of the augment bar, so $k + 12 \neq 0$ and h can be anything.

$$\boxed{h \text{ any real number}, \qquad k \neq -12}.$$

Free response. Show your work! A correct answer without appropriate work will receive little or no credit, even if the answer is correct.

6. (10 points) A baker makes two types of cake. Each type of cake requires a certain amount of flour, butter and sugar.

- The first type is a chocolate lava cake that contains 1 ounce of sugar, 2 ounces of flour, and 4 ounces of butter.
- The second type is a carrot cake that contains 2 ounces of sugar, 3 ounces of flour, and 7 ounces of butter.

The baker has 14 ounces of sugar, 26 ounces of flour, and 54 ounces of butter. Let x be the number of chocolate lava cakes and y be the number of carrot cakes that the baker can make using their full supply of sugar, flour, and butter.

This problem is a slight modification of #5 from the 1.2 Webwork.

a) Write a system of linear equations that we could solve in order to find x and y.

Solution: We'll put the equations along with the ingredients.

Sugar: $x + 2y = 14$
Flour: $2x + 3y = 26$
Butter: $4x + 7y = 54$.

b) Write a vector equation that we could solve in order to find x and y.

Solution: Representing part (a) with vectors gives us

$$x\begin{pmatrix}1\\2\\4\end{pmatrix} + y\begin{pmatrix}2\\3\\7\end{pmatrix} = \begin{pmatrix}14\\26\\54\end{pmatrix}.$$

c) Solve for x and y by putting an augmented matrix into reduced row echelon form. Please note that in order to receive full credit, you must write an augmented matrix and put it into RREF. If you simply guess and check, you will receive little or no credit, even if your answer is correct. Enter your answer below.

$\boxed{x = 10 \qquad y = 2}$

Solution:

$$\left(\begin{array}{cc|c}1 & 2 & 14\\2 & 3 & 26\\4 & 7 & 54\end{array}\right) \xrightarrow[R_3=R_3-4R_1]{R_2=R_2-2R_1} \left(\begin{array}{cc|c}1 & 2 & 14\\0 & -1 & -2\\0 & -1 & -2\end{array}\right) \xrightarrow[\text{then } R_3=R_3+R_2]{R_2=-R_2} \left(\begin{array}{cc|c}1 & 2 & 14\\0 & 1 & 2\\0 & 0 & 0\end{array}\right) \xrightarrow{R_1=R_1-2R_2} \left(\begin{array}{cc|c}1 & 0 & 10\\0 & 1 & 2\\0 & 0 & 0\end{array}\right).$$

Therefore $x = 10$ and $y = 2$.

This problem was taken almost directly from #6 of Sample Midterm 1A and #5 of Sample Midterm 1B.

7. Consider the following linear system of equations in the variables x_1, x_2, x_3, x_4:

$$\begin{aligned} x_1 + x_2 - 2x_3 + x_4 &= 3 \\ 3x_1 + 4x_2 + x_3 + 3x_4 &= 4 \\ -2x_1 - 2x_2 + 4x_3 - x_4 &= -8. \end{aligned}$$

a) (5 points) Write the augmented matrix corresponding to this system, and put the augmented matrix into RREF.

Solution:

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 3 \\ 3 & 4 & 1 & 3 & 4 \\ -2 & -2 & 4 & -1 & -8 \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2-3R_1} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 1 & 3 \\ 0 & 1 & 7 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right) \xrightarrow{R_1=R_1-R_3} \left(\begin{array}{cccc|c} 1 & 1 & -2 & 0 & 5 \\ 0 & 1 & 7 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-R_2} \left(\begin{array}{cccc|c} 1 & 0 & -9 & 0 & 10 \\ 0 & 1 & 7 & 0 & -5 \\ 0 & 0 & 0 & 1 & -2 \end{array}\right)$$

b) (4 points) The system is consistent. Write the set of solutions to the system of equations in parametric vector form.

Solution: We get $x_1 - 9x_3 = 10$, $x_2 + 7x_3 = -5$, and $x_4 = -2$ with x_3 free, so the parametric form is

$$x_1 = 10 + 9x_3, \qquad x_2 = -5 - 7x_3, \qquad x_3 = x_3 \quad x_3 \text{ (real)}, \qquad x_4 = -2.$$

This gives

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 10 + 9x_3 \\ -5 - 7x_3 \\ x_3 \\ -2 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 0 \\ -2 \end{pmatrix} + \begin{pmatrix} 9x_3 \\ -7x_3 \\ x_3 \\ 0 \end{pmatrix} = \begin{pmatrix} 10 \\ -5 \\ 0 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 9 \\ -7 \\ 1 \\ 0 \end{pmatrix}.$$

c) (1 points) Write **one** vector x that solves the linear system of equations. There is no partial credit on this part, so take time to check by hand that your answer is correct, and if it is not correct then go back and check your work from above!

Solution: $x = \begin{pmatrix} 10 \\ -5 \\ 0 \\ -2 \end{pmatrix}$ or $x = \begin{pmatrix} 10+9 \\ -5-7 \\ 0+1 \\ -2+0 \end{pmatrix} = \begin{pmatrix} 19 \\ -12 \\ 1 \\ -2 \end{pmatrix}$ (when $x_3 = 1$) are possible answers, and in fact we can take any value for x_3 and compute x using the answer from part (b). Note that to get credit on this part you **must** write a **correct answer**, you cannot just write an answer that follows your work from (b). This is because the instructions were very specific to check your answer to make sure it solved the system of equations, and that if you found your answer was incorrect then you needed to go back and check your work in (a) and (b) to see where you went wrong.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.