MATH 1553, EXAM 2 SOLUTIONS
SPRING 2024

Name		GT ID	

Circle your instructor and lecture below. Some professors teach more than one lecture, so be sure to circle the correct choice!

Jankowski (A and HP, 8:25-9:15 AM) Jankowski (G, 12:30-1:20 PM)

Hausmann (I, 2:00-2:50 PM) Sanchez-Vargas (M, 3:30-4:20 PM)

Athanasouli (N and PNA, 5:00-5:50 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- Good luck!

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 7:45 PM on Wednesday, March 6.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

a) Suppose v_1, v_2, and b are vectors in $\mathbf{R}^n$ and the equation

$$x_1v_1 + x_2v_2 = b$$

has exactly one solution. Then $\{v_1, v_2, b\}$ must be linearly independent.

TRUE $\boxed{\text{FALSE}}$

b) Let V be the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ of the form

$$x - 3y + z = 2.$$

Then V is a subspace of $\mathbf{R}^3$.

TRUE $\boxed{\text{FALSE}}$

c) If a vector x is in the null space of a matrix A, then $4x$ is also in the null space of A.

$\boxed{\text{TRUE}}$ FALSE

d) If A is a 4×3 matrix, then the matrix transformation $T(x) = Ax$ cannot be onto.

$\boxed{\text{TRUE}}$ FALSE

e) If A is a 3×3 matrix and the equation $Ax = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ has exactly one solution, then A must be invertible.

$\boxed{\text{TRUE}}$ FALSE

Solution.

a) False. The fact that $x_1v_1+x_2v_2 = b$ is consistent means that b is in Span$\{v_1, v_2\}$, so the set $\{v_1, v_2, b\}$ is automatically linearly dependent by the Increasing Span Criterion.

b) False. If we check the conditions for being a subspace, we find that V immediately fails the first property: V does not contain $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ since

$$0 - 3(0) + 0 \neq 2.$$

In reality, V is a plane in $\mathbf{R}^3$ but it does not have any of the three properties of a subspace.

c) True: if A is any $m \times n$ matrix then Nul A is a subspace of $\mathbf{R}^n$ and therefore Nul A is closed under scalar multiplication, so if x is in Nul A then so is every scalar multiple of x. Alternatively, we could have just used computations rather than the theory of subspaces. If x is in the null space of A, then $Ax = 0$, so $A(4x) = 4Ax = 4(0) = 0$, therefore $4x$ is also in the null space of A.

d) True: the 4×3 matrix A has 4 rows but a maximum of 3 pivots, therefore A will have at least one row without a pivot and so T cannot be onto.

e) True, by the Invertible Matrix Theorem. If $Ax = b$ has a unique solution for some b, then $Ax = 0$ has exactly one solution, which means that the $n \times n$ matrix A has a pivot in every column and is therefore invertible.

2. **Full solutions are on the next page.**

a) (3 points) Suppose v_1, v_2, v_3, and v_4 are vectors in $\mathbf{R}^4$. Which of the following must be true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ If $\mathrm{Span}\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4$, then the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

$\boxed{\text{(ii)}}$ Suppose that the matrix A with columns v_1 through v_4 has one pivot, and let T be the matrix transformation $T(x) = Ax$. Then the range of T is a line.

$\boxed{\text{(iii)}}$ If $v_1 - v_2 - v_3 + v_4 = 0$, then $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

b) (4 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \ : \ y \leq 2x \right\}$.

Answer each of the following questions.

(i) Does V contain the zero vector? $\boxed{\text{YES}}$ NO

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? $\boxed{\text{YES}}$ NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? YES $\boxed{\text{NO}}$

(iv) Is there a matrix A with the property that $\mathrm{Col}(A) = V$? YES $\boxed{\text{NO}}$

c) (3 pts) Let A be a 15×7 matrix. If $\mathrm{rank}(A) \leq 3$, which of the following are possible values for $\mathrm{nullity}(A)$? Clearly circle **all** that apply.

(i) 1

(ii) 3

$\boxed{\text{(iii)}}$ 4

$\boxed{\text{(iv)}}$ 6

$\boxed{\text{(v)}}$ 7

(vi) 9

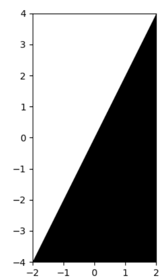
Solution to Problem 2.

a) (i) True by the Basis Theorem: if the span of the 4 vectors is $\mathbf{R}^4$, then the vectors are automatically linearly independent. Alternatively, we could reason without using the Basis Theorem. if the span of those 4 vectors is $\mathbf{R}^4$, then the matrix with columns v_1 through v_4 will have 4 pivots, thus A has a pivot in every column and so the vectors are linearly independent.

(ii) True: the range of T is the column space of A, which is a line since A has one pivot.

(iii) True directly by the definition of linearly independence. In fact, the equation $v_1 - v_2 - v_3 + v_4 = 0$ is a linear dependence relation for the vectors.

b) This problem was nearly copied from #2a from Sample Midterm 2A, and #7a from Sample Midterm 2B. We can draw V and see right away that it is not a subspace of $\mathbf{R}^2$, since it is neither the zero vector nor a line through the origin nor the entirety of $\mathbf{R}^2$. The set V consists of all points on, and below, the line $y = 2x$.



(i) Yes: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ satisfies $y \leq 2x$ since $0 \leq 0$.

(ii) Yes: we can see geometrically that if we add any vectors in the region V, the sum will stay in V. We could have done it algebraically instead, those this is not as easy. If $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are in V (so $y_1 \leq 2x_1$ and $y_2 \leq 2x_2$) then $\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ is in V since

$$y_1 + y_2 \leq 2x_1 + 2x_2 \leq 2(x_1 + x_2).$$

(iii) No. We can see geometrically that if u is a nonzero vector in V, then $-u$ is often not in V. Alternatively, we can see this geometrically. For example, take $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We see u is in V since $0 \leq 2(1)$. However, $-u = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is not in V since $0 > -2$.

(iv) No: column spaces are always subspaces, but V is not a subspace, so V cannot be the column space of any matrix. Alternatively, without even doing (i)-(iii) we could do this using spans: if V were the span of the columns of some matrix, then since V contains $(1, 0)$ and $(0, -1)$ (since they satisfy $y \leq 2x$) it would need contain their span, which is all of $\mathbf{R}^2$. But this is nonsense, since V is clearly not $\mathbf{R}^2$.

c) This is a classic Rank Theorem problem. By the Rank Theorem we know

$$\text{rank}(A) + \text{nullity}(A) = 7, \qquad \text{so} \qquad \text{nullity}(A) = 7 - \text{rank}(A).$$

From the fact that $0 \leq \text{rank}(A) \leq 3$, we know that nullity$(A)$ must be at least 4 but cannot be greater than 7.

3. **Full solutions are on the next page.**

a) (2 pts) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation for **counterclockwise** rotation by 13°.

(i) Find the standard matrix A for the transformation T.

$\boxed{\text{(I) } A = \begin{pmatrix} \cos(13^\circ) & -\sin(13^\circ) \\ \sin(13^\circ) & \cos(13^\circ) \end{pmatrix}}$ (II) $A = \begin{pmatrix} \cos(13^\circ) & \sin(13^\circ) \\ -\sin(13^\circ) & \cos(13^\circ) \end{pmatrix}$

(III) $A = \begin{pmatrix} \cos(13^\circ) & -\sin(13^\circ) \\ -\sin(13^\circ) & \cos(13^\circ) \end{pmatrix}$ (IV) $A = \begin{pmatrix} -\cos(13^\circ) & -\sin(13^\circ) \\ \sin(13^\circ) & -\cos(13^\circ) \end{pmatrix}$

(ii) T is invertible. Find the standard matrix B for the transformation T^{-1}.

(I) $B = \begin{pmatrix} \cos(13^\circ) & -\sin(13^\circ) \\ \sin(13^\circ) & \cos(13^\circ) \end{pmatrix}$ $\boxed{\text{(II) } B = \begin{pmatrix} \cos(13^\circ) & \sin(13^\circ) \\ -\sin(13^\circ) & \cos(13^\circ) \end{pmatrix}}$

(III) $B = \begin{pmatrix} \cos(13^\circ) & -\sin(13^\circ) \\ -\sin(13^\circ) & \cos(13^\circ) \end{pmatrix}$ (IV) $B = \begin{pmatrix} -\cos(13^\circ) & -\sin(13^\circ) \\ \sin(13^\circ) & -\cos(13^\circ) \end{pmatrix}$

b) (3 points) Let A be an $m \times n$ matrix, and let T be the associated linear transformation $T(x) = Ax$. Which of the following statements are true? Clearly circle all that apply.

$\boxed{\text{(i)}}$ If the columns of A are linearly independent, then $\dim(\text{range}(T)) = n$.

(ii) If the columns of A are linearly dependent, then T is onto.

$\boxed{\text{(iii)}}$ If T is onto, then $\text{Col}(A) = \mathbf{R}^m$.

c) (3 points) Determine which of the following statements are true.

$\boxed{\text{(i)}}$ If A and B are invertible $n \times n$ matrices, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

(ii) If A is a 3×7 matrix and B is a 4×3 matrix, then the matrix transformation T given by $T(x) = BAx$ has domain $\mathbf{R}^4$ and codomain $\mathbf{R}^7$.

$\boxed{\text{(iii)}}$ If A is an $n \times n$ matrix and $A^2 = 0$, then $(I - A)(I + A) = I$.

d) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first rotates vectors by 90 degrees clockwise, then reflects across the line $y = -x$. Find $T\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Clearly circle your answer below.

(i) $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ $\boxed{\text{(ii) } \begin{pmatrix} 0 \\ -2 \end{pmatrix}}$ (iii) $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ (iv) $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$ (v) $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ (vi) $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution to Problem 3.

a) This is a quintessential rotations problem we have seen many times before.

(i) Counterclockwise rotation by angle θ is given by $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, so the answer is (I): $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ \sin(13°) & \cos(13°) \end{pmatrix}$

(ii) The inverse is rotation clockwise by angle θ, so the answer is (II):

$$B = \begin{pmatrix} \cos(13°) & \sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}.$$

b) (i) True: if the columns of A are linearly independent, then all n columns have pivots and are therefore pivot columns, so $\dim(\mathrm{Col}(A)) = n$.

(ii) Not necessarily true. For example, take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(iii) True: in fact this is nearly one of the definitions of "onto."

c) This problem has an emphasized key fact about inverses, along with basics of multiplication and a slight modification of #4 from the 3.5+3.6 Webwork.

(i) True, a standard fact from 3.5+3.6.

(ii) Not true: BA will be a 4×7 matrix, so T will have domain $\mathbf{R}^7$ and codomain $\mathbf{R}^4$.
(iiI) True:

$$(I - A)(I + A) = I^2 + A - A - A^2 = I - A^2 = I - 0 = I.$$

d) We rotate $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ clockwise by 90 degrees to get $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, then we reflect $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ across the line $y = -x$ to get $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. Alternatively, we could do matrix multiplication instead. Since we are doing the rotation first and then the reflection second, we must put the rotation matrix on the right and the reflection on the left (this is not a typo):

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so our answer is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

4. **a)** (4 points) Suppose A is a 2000×30 matrix and its RREF has 20 pivots. Answer each of the following questions.

(i) Fill in the blank: the rank of A is $\underline{\ 20\ }$

(ii) Fill in the blank: the nullity of A is $\underline{\ 10\ }$

(iii) Circle the correct answer below to complete the following sentence:
Col(A) is a subspace of...

$\mathbf{R}^{10}$ $\quad$ $\mathbf{R}^{20}$ $\quad$ $\mathbf{R}^{30}$ $\quad$ $\mathbf{R}^{1980}$ $\quad$ $\mathbf{R}^{1990}$ $\quad$ $\boxed{\mathbf{R}^{2000}}$

(iv) Circle the correct answer below to complete the following sentence:
Nul(A) is a subspace of...

$\mathbf{R}^{10}$ $\quad$ $\mathbf{R}^{20}$ $\quad$ $\boxed{\mathbf{R}^{30}}$ $\quad$ $\mathbf{R}^{1980}$ $\quad$ $\mathbf{R}^{1990}$ $\quad$ $\mathbf{R}^{2000}$

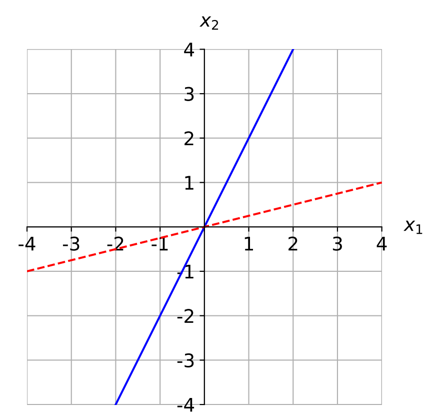
b) (4 pts) Write a matrix A so that Col(A) is the solid line below and Nul(A) is the dashed line below.

Many answers possible, for example $A = \begin{pmatrix} 1 & -4 \\ 2 & -8 \end{pmatrix}$.

In order for Col(A) to be the span of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, each column must be a scalar multiple of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, and at least one column must be nonzero.

Also, since the null space is the span of $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$, we know the second column must be -4 times the first column so that $A\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Putting this together, we get many correct answers such as

$$A = \begin{pmatrix} 1 & -4 \\ 2 & -8 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -8 \\ 4 & -16 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 4 \\ -2 & 8 \end{pmatrix}, \quad \text{etc.}$$



c) (2 pts) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 11 \end{pmatrix}$. Find A^{-1}. Clearly circle your answer below.

(i) $A^{-1} = \frac{1}{5}\begin{pmatrix} 11 & 2 \\ 3 & 1 \end{pmatrix}$ (ii) $A^{-1} = \frac{1}{17}\begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix}$ (iii) $A^{-1} = \frac{1}{5}\begin{pmatrix} 11 & -3 \\ -2 & 1 \end{pmatrix}$

$\boxed{\text{(iv) } A^{-1} = \frac{1}{5}\begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix}}$ (v) $A^{-1} = \begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix}$ (vi) $A^{-1} = \frac{1}{5}\begin{pmatrix} 1 & -2 \\ -3 & 11 \end{pmatrix}$

Solution: Using the notation $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we get

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{1(11)-3(2)}\begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 11 & -2 \\ -3 & 1 \end{pmatrix}.$$

This problem is mostly a combination of #5ab from Sample Midterm 2A and #6b from Sample Midterm 2B.

5. Consider the following matrix A below in its reduced row echelon form, and let T be the matrix transformation $T(x) = Ax$.

$$A = \begin{pmatrix} 1 & 1 & -3 & 2 \\ 1 & -1 & -1 & 0 \\ 1 & 3 & -5 & 4 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col(A). You don't need to show your work on this part.

Solution: The pivot columns of A are guaranteed to form a basis for Col A, so a correct answer is $\left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix} \right\}$. However, in this case, *any two columns* of A will automatically form a basis for Col A.

b) Write a new basis for Col(A), so that no vector in your new basis is a scalar multiple of any of the vectors in the basis you wrote in part (a). Clearly show how you obtain this new basis.

Solution: Many answers possible. If we let v_1 and v_2 be the basis from (a), then a new basis would be $\{v_1 - v_2, v_1 + v_2\}$ which is $\left\{ \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 4 \end{pmatrix} \right\}$.

c) Find a basis for Nul(A).

Solution: From the RREf of A we know that the RREF of $(A|0)$ yields $x_1 - 2x_3 + x_4 = 0$ and $x_2 - x_3 + x_4 = 0$ with x_3 and x_4 are free. Therefore $x_1 = 2x_3 - x_4$ and $x_2 = x_3 - x_4$, so

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2x_3 - x_4 \\ x_3 - x_4 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}. \qquad \text{Basis}: \left\{ \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

d) Write **one** solution x to the equation $T(x) = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$. There is no partial credit on this part, so take time to check by hand that your answer is correct.

Solution: $A = \begin{pmatrix} T(e_1) & T(e_2) & T(e_3) & T(e_4) \end{pmatrix}$ and the 2nd column of A is $\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$, we know without any work that $T(e_2) = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}$. Therefore, an answer is $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. In reality, the sum of $\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$ and any vector in the null space of A will be a correct answer, for example

$$\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}2\\1\\1\\0\end{pmatrix}=\begin{pmatrix}2\\2\\1\\0\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}=\begin{pmatrix}-1\\0\\0\\1\end{pmatrix} \quad\text{or}\quad \begin{pmatrix}0\\1\\0\\0\end{pmatrix}-\begin{pmatrix}-1\\-1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\2\\0\\-1\end{pmatrix}.$$

Free response. Show your work! A correct answer without appropriate work will receive little or no credit, even if the answer is correct.

6. Let $T : \mathbf{R}^2 \to \mathbf{R}^3$ be the transformation $T(x_1, x_2) = (3x_1 - x_2,\ 5x_2 - x_1, x_1 + x_2)$. Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects vectors across the line $y = x$.

a) (3 points) Find the standard matrix A for T.

$$A = \left(T\begin{pmatrix}1\\0\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix} 3 & -1 \\ -1 & 5 \\ 1 & 1 \end{pmatrix}.$$

b) (2 points) Write the standard matrix B for U.

$$B = \left(T\begin{pmatrix}1\\0\end{pmatrix} \quad T\begin{pmatrix}0\\1\end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

c) (1 point) Which **one** of the following compositions makes sense? (no work required)

$\boxed{T \circ U}$ $\qquad\qquad$ $U \circ T$

d) (4 points) For the composition you circled in part (c), compute the standard matrix for the transformation.

The matrix for $T \circ U$ is AB, and we compute

$$AB = \begin{pmatrix} 3 & -1 \\ -1 & 5 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 3 \\ 5 & -1 \\ 1 & 1 \end{pmatrix}$$

7. Free response. Show your work in (b) and (c)! Parts (a), (b), and (c) are unrelated.

a) (2 points) Suppose A is a matrix and T is the corresponding matrix transformation $T(x) = Ax$. Fill in the blanks below. You do not need to show your work on this part.

(i) T is one-to-one if A has a pivot in every __column__.

(ii) T is onto if A has a pivot in every __row__.

The question does not say how many rows or columns A has, so we cannot accept "m" or "n" as an answer (they also would not make any grammatical sense) and we cannot assume that A is a square matrix.

b) (4 points) Let $A = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}$. Find a nonzero matrix B so that AB is the 2×2 zero matrix. Enter your answer in the space provided below.

Solution: We solve for $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ so that $AB = 0$.

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & -6 \\ 1 & -2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 3a-6c & 3b-6d \\ a-2c & b-2d \end{pmatrix}.$$

The above is equivalent to just $a - 2c = 0$ and $b - 2d = 0$, so $a = 2c$ and $b = 2d$. In other words, the first row just needs to be twice the second row.

This means that each column must be a multiple of $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and at least one column must not the zero vector (otherwise B would be the zero matrix). We can have

$B = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 2 \\ 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 2 \\ -1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1/2 & 0 \end{pmatrix}$, etc..

c) (4 points) Find all real values of c (if there are any) so that the following set is linearly independent.

$$\left\{ \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} 2 \\ c \\ -4 \end{pmatrix} \right\}$$

Solution: We put the vectors as columns of a matrix and row-reduce. The set is linearly independent if and only if the matrix has 3 pivots.

$$\begin{pmatrix} 1 & 1 & 2 \\ 3 & -1 & c \\ -1 & 3 & -4 \end{pmatrix} \xrightarrow[R_3 = R_3 + R_1]{R_2 = R_2 - 3R_1} \begin{pmatrix} 1 & 1 & 2 \\ 0 & -4 & c-6 \\ 0 & 4 & -2 \end{pmatrix} \xrightarrow{R_3 = R_3 + R_2} \begin{pmatrix} \boxed{1} & 1 & 2 \\ 0 & \boxed{-4} & c-6 \\ 0 & 0 & c-8 \end{pmatrix}.$$

The matrix automatically has at least 2 pivots, and it will have 3 pivots precisely when the bottom right entry is nonzero. Therefore $c - 8 \neq 0$, so $c \neq 8$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.