Supplemental problems: §5.5

1. **a)** If A is the matrix that implements rotation by $143°$ in $\mathbf{R}^2$, then A has no real eigenvalues.

b) A 3×3 matrix can have eigenvalues 3, 5, and $2+i$.

c) If $v = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$, then $w = \begin{pmatrix} 2i-1 \\ i \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = 1-i$.

Solution.

a) True. If A had a real eigenvalue λ, then we would have $Ax = \lambda x$ for some nonzero vector x in $\mathbf{R}^2$. This means that x would lie on the same line through the origin as the rotation of x by $143°$, which is impossible.

b) False. If $2+i$ is an eigenvalue then so is its conjugate $2-i$.

c) True. Any nonzero complex multiple of v is also an eigenvector for eigenvalue $1-i$, and $w = iv$.

2. Consider the matrix

$$A = \begin{pmatrix} 3\sqrt{3}-1 & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1 \end{pmatrix}$$

a) Find both complex eigenvalues of A.

b) Find an eigenvector corresponding to each eigenvalue.

Solution.

a) We compute the characteristic polynomial:

$$\begin{aligned} f(\lambda) &= \det\begin{pmatrix} 3\sqrt{3}-1-\lambda & -5\sqrt{3} \\ 2\sqrt{3} & -3\sqrt{3}-1-\lambda \end{pmatrix} \\ &= (-1-\lambda+3\sqrt{3})(-1-\lambda-3\sqrt{3})+(2)(5)(3) \\ &= (-1-\lambda)^2-9(3)+10(3) \\ &= \lambda^2+2\lambda+4. \end{aligned}$$

By the quadratic formula,

$$\lambda = \frac{-2\pm\sqrt{2^2-4(4)}}{2} = \frac{-2\pm 2\sqrt{3}i}{2} = -1\pm\sqrt{3}i.$$

b) Let $\lambda = -1-\sqrt{3}i$. Then

$$A-\lambda I = \begin{pmatrix} (i+3)\sqrt{3} & -5\sqrt{3} \\ 2\sqrt{3} & (i-3)\sqrt{3} \end{pmatrix}.$$

Since $\det(A-\lambda I)=0$, the second row is a multiple of the first, so a row echelon form of A is

$$\begin{pmatrix} i+3 & -5 \\ 0 & 0 \end{pmatrix}.$$

Hence an eigenvector with eigenvalue $-1-\sqrt{3}i$ is $v=\begin{pmatrix} 5 \\ 3+i \end{pmatrix}$. It follows that an eigenvector with eigenvalue $-1+\sqrt{3}i$ is $\overline{v}=\begin{pmatrix} 5 \\ 3-i \end{pmatrix}$.

3. This problem is an example of a 3×3 matrix that has a mix of real and (non-real) complex eigenvalues. In such a case, we are not able to use the "2×2 eigenvector trick" because the matrix is 3×3, and so we would need to do row-reduction to find the complex eigenvectors. This particular kind of problem is computationally beyond the scope of what to expect on a Math 1553 exam in Spring 2022.

Let $A=\begin{pmatrix} 4 & -3 & 3 \\ 3 & 4 & -2 \\ 0 & 0 & 2 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue of A, find a corresponding eigenvector.

Solution.

First we compute the characteristic polynomial by expanding cofactors along the third row:

$$f(\lambda)=\det\begin{pmatrix} 4-\lambda & -3 & 3 \\ 3 & 4-\lambda & -2 \\ 0 & 0 & 2-\lambda \end{pmatrix}=(2-\lambda)\det\begin{pmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{pmatrix}$$
$$=(2-\lambda)\big((4-\lambda)^2+9\big)=(2-\lambda)(\lambda^2-8\lambda+25).$$

Using the quadratic equation on the second factor, we find the eigenvalues

$$\lambda_1=2 \qquad \lambda_2=4-3i \qquad \overline{\lambda_2}=4+3i.$$

Next compute an eigenvector with eigenvalue $\lambda_1=2$:

$$A-2I=\begin{pmatrix} 2 & -3 & 3 \\ 3 & 2 & -2 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The parametric form is $x=0$, $y=z$, so the parametric vector form of the solution is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}=z\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_1=\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}.$$

Now we compute an eigenvector with eigenvalue $\lambda_2 = 4-3i$:

$$A = (4-3i)I = \begin{pmatrix} 3i & -3 & 3 \\ 3 & 3i & -2 \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_1 \longleftrightarrow R_2} \begin{pmatrix} 3 & 3i & -2 \\ 3i & -3 & 3 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{R_2 = R_2 - iR_1} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 3+2i \\ 0 & 0 & 3i-2 \end{pmatrix} \xrightarrow{R_2 = R_2 \div (3+2i)} \begin{pmatrix} 3 & 3i & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 3i-2 \end{pmatrix}$$

$$\xrightarrow{\text{row replacements}} \begin{pmatrix} 3 & 3i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_1 = R_1 \div 3} \begin{pmatrix} 1 & i & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

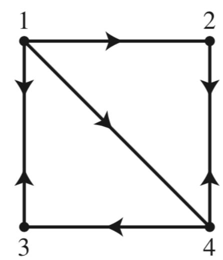
The parametric form of the solution is $x = -iy$, $z = 0$, so the parametric vector form is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix} \xrightarrow{\text{eigenvector}} v_2 = \begin{pmatrix} -i \\ 1 \\ 0 \end{pmatrix}.$$

An eigenvector for the complex conjugate eigenvalue $\overline{\lambda}_2 = 4+3i$ is the complex conjugate eigenvector $\overline{v}_2 = \begin{pmatrix} i \\ 1 \\ 0 \end{pmatrix}$.

Supplemental problems: §5.6

1. Suppose the internet has four pages in the following manner. Arrows represent links from one page towards another. For example, page 1 links to page 4 but not vice versa.



a) Write the importance matrix and the Google matrix for this internet using damping constant $p = 0.15$. You don't need to simplify the Google matrix.

b) The steady-state vector for the Google matrix is (approximately)

$$\begin{pmatrix} 0.23 \\ 0.23 \\ 0.23 \\ 0.31 \end{pmatrix}.$$

What is the top-ranked page?

Solution.

(a) The importance matrix is

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{pmatrix}$$

The Google matrix is

$$(1-p)A + pB$$

$$0.85 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 0 & 0 & 1/2 \\ 1/3 & 1 & 0 & 0 \end{pmatrix} + (0.15)\frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}.$$

(b) From the steady-state vector we see page 4 has the highest rank.

2. The companies X, Y, and Z fight for customers. This year, company X has 40 customers, Company Y has 15 customers, and Z has 20 customers. Each year, the following changes occur:

- X keeps 75% of its customers, while losing 15% to Y and 10% to Z.
- Y keeps 60% of its customers, while losing 5% to X and 35% to Z.

- Z keeps 65% of its customers, while losing 15% to X and 20% to Y.

Write a stochastic matrix A and a vector x so that Ax will give the number of customers for firms X, Y, and Z (respectively) after one year. You do not need to compute Ax.

Solution.

$$A = \begin{pmatrix} 0.75 & 0.05 & 0.15 \\ 0.15 & 0.6 & 0.20 \\ 0.1 & 0.35 & 0.65 \end{pmatrix} \qquad x = \begin{pmatrix} 40 \\ 15 \\ 20 \end{pmatrix}.$$

3. Courage Soda and Dexter Soda compete for a market of 210 customers who drink soda each day.

Today, Courage has 80 customers and Dexter has 130 customers. Each day:

70% of Courage Soda's customers keep drinking Courage Soda, while 30% switch to Dexter Soda.

40% of Dexter Soda's customers keep drinking Dexter Soda, while 60% switch to Courage Soda.

a) Write a stochastic matrix A and a vector x so that Ax will give the number of customers for Courage Soda and Dexter Soda (in that order) tomorrow.
You do not need to compute Ax.

$$A = \begin{pmatrix} 0.7 & 0.6 \\ 0.3 & 0.4 \end{pmatrix} \text{ and } x = \begin{pmatrix} 80 \\ 130 \end{pmatrix}.$$

b) Find the steady-state vector for A.

$$\left(A - I \mid 0\right) = \left(\begin{array}{cc|c} -0.3 & 0.6 & 0 \\ 0.3 & -0.6 & 0 \end{array}\right) \xrightarrow[R_1 = R_1/(-0.3)]{R_2 = R_2 + R_1} \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$$

so $x_1 = 2x_2$ and x_2 is free. A 1-eigenvector is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, so the steady state vector is $w = \frac{1}{2+1}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix}$.

c) Use your answer from (b) to determine the following: in the long run, roughly how many daily customers will Courage Soda have?

As n gets large, $A^n \begin{pmatrix} 80 \\ 130 \end{pmatrix}$ approaches $210 \begin{pmatrix} 2/3 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 140 \\ 70 \end{pmatrix}$. Courage will have roughly 140 customers.