Supplemental problems: §3.2

1. Let A be a 3×4 matrix with column vectors v_1, v_2, v_3, v_4, and suppose $v_2 = 2v_1 - 3v_4$. Consider the matrix transformation $T(x) = Ax$.

a) Is it possible that T is one-to-one? If yes, justify why. If no, find distinct vectors v and w so that $T(v) = T(w)$.

b) Is it possible that T is onto? Justify your answer.

2. **a)** Which of the following are onto transformations? (Check all that apply.)

- ☐ $T : \mathbf{R}^3 \to \mathbf{R}^3$, reflection over the xy-plane
- ☐ $T : \mathbf{R}^3 \to \mathbf{R}^3$, projection onto the xy-plane
- ☐ $T : \mathbf{R}^3 \to \mathbf{R}^2$, project onto the xy-plane, forget the z-coordinate
- ☐ $T : \mathbf{R}^2 \to \mathbf{R}^2$, scale the x-direction by 2

b) Let A be a square matrix (square means $n \times n$) and let $T(x) = Ax$. Which of the following guarantee that T is onto? (Check all that apply.)

- ☐ T is one-to-one
- ☐ $Ax = 0$ is consistent

3. Which of the following transformations are **one-to-one**? Circle all that apply.

(i) $T : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $T(x,y) = (x-y,\ x-2y,\ x+y)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates vectors counterclockwise by 17 degrees.

(iii) The transformation $T(x) = Ax$, where $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$.

4. Which of the following transformations are **onto**? Circle all that apply.

(i) $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x,y,z) = (x+y,\ x+y)$.

(ii) $T : \mathbf{R}^2 \to \mathbf{R}^2$ which rotates vectors counterclockwise by 17 degrees.

(iii) The transformation $T(x) = Ax$, where A is a 3×3 matrix with the property that the equation $Av = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ has exactly one solution.

5. Find all real numbers h so that the transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ given by

$$T(v) = \begin{pmatrix} -1 & 0 & 2-h \\ h & 0 & 3 \end{pmatrix} v$$

is onto.

Supplemental problems: §3.3

1. Circle **T** if the statement is always true, and circle **F** otherwise.

a) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^n$ is linear and $T(e_1) = T(e_2)$, then the homogeneous equation $T(x) = 0$ has infinitely many solutions.

b) **T** **F** If $T : \mathbf{R}^n \to \mathbf{R}^\mathbf{m}$ is a one-to-one linear transformation and $m \neq n$, then T must not be onto.

2. Consider $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by

$$T(x, y, z) = (x,\ x + z,\ 3x - 4y + z,\ x).$$

Is T one-to-one? Justify your answer.

3. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the matrix is not one-to-one, find two vectors with the same image.

a) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (y, y)$.

b) JUST FOR FUN: Consider T : (Smooth functions) $\to$ (Smooth functions) given by $T(f) = f'$ (the derivative of f). Then T is not a transformation from any $\mathbf{R}^n$ to $\mathbf{R}^m$, but it is still *linear* in the sense that for all smooth f and g and all scalars c (by properties of differentiation we learned in Calculus 1):

$$T(f + g) = T(f) + T(g) \quad \text{since} \quad (f + g)' = f' + g'$$
$$T(cf) = cT(f) \quad \text{since} \quad (cf)' = cf'.$$

Is T one-to-one?

4. In each case, determine whether T is linear. Briefly justify.

a) $T(x_1, x_2) = (x_1 - x_2, x_1 + x_2, 1)$.

b) $T(x, y) = (y, x^{1/3})$.

c) $T(x, y, z) = 2x - 5z$.

5. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis (look downward onto the xy-plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane.

In the worksheet, we found the matrix for the transformation T caused by the wolf. Geometrically describe the image of the house under T.