Supplemental problems: §2.3 and §2.4

The professor in your widgets and gizmos class is trying to decide between three different grading schemes for computing your final course grade. The schemes are based on homework (HW), quiz grades (Q), midterms (M), and a final exam (F). The three schemes can be described by the following matrix A:

$$\begin{array}{r} \\ \text{Scheme 1} \\ \text{Scheme 2} \\ \text{Scheme 3} \end{array} \begin{array}{c} \begin{array}{cccc} \text{HW} & \text{Q} & \text{M} & \text{F} \end{array} \\ \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \end{array}$$

1. Suppose that you have a score of x_1 on homework, x_2 on quizzes, x_3 on midterms, and x_4 on the final, with potential final course grades of b_1, b_2, b_3.

a) Write a matrix equation $Ax = b$ to relate your final grades to your scores.

b) Row reduce the corresponding augmented matrix until you reach reduced row echelon form.

c) Looking at the final matrix in (b), what equation in terms of b_1, b_2, b_3 must be satisfied in order for $Ax = b$ to have a solution?

d) The answer to (c) also defines the span of the columns of A. Describe the span geometrically.

e) Solve the equation in (c) for b_1. Looking at this equation, is it possible for b_1 to be the largest of b_1, b_2, b_3? In other words, is it ever possible for the grade under Scheme 1 to be the highest of the three final course grades? Why or why not? Which scheme would you argue for?

Solution.

a) In the above grading schemes, you would receive the following final grades:

$$\begin{array}{ll} \text{Scheme 1:} & 0.1x_1 + 0.1x_2 + 0.5x_3 + 0.3x_4 = b_1 \\ \text{Scheme 2:} & 0.1x_1 + 0.1x_2 + 0.4x_3 + 0.4x_4 = b_2 \\ \text{Scheme 3:} & 0.1x_1 + 0.1x_2 + 0.6x_3 + 0.2x_4 = b_3 \end{array}$$

This is the same as the matrix equation

$$(*) \qquad \begin{pmatrix} 0.1 & 0.1 & 0.5 & 0.3 \\ 0.1 & 0.1 & 0.4 & 0.4 \\ 0.1 & 0.1 & 0.6 & 0.2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}.$$

b) Here is the row reduction:

$$\left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0.1 & 0.1 & 0.4 & 0.4 & b_2 \\ 0.1 & 0.1 & 0.6 & 0.2 & b_3 \end{array}\right) \xrightarrow[R_3 = R_3 - R_1]{R_2 = R_2 - R_1} \left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0 & 0 & -0.1 & 0.1 & b_2 - b_1 \\ 0 & 0 & 0.1 & -0.1 & b_3 - b_1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 + R_2} \left(\begin{array}{cccc|c} 0.1 & 0.1 & 0.5 & 0.3 & b_1 \\ 0 & 0 & -0.1 & 0.1 & b_2 - b_1 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right)$$

$$\xrightarrow[R_2 = R_2 \times (-10)]{R_1 = R_1 \times 10} \left(\begin{array}{cccc|c} 1 & 1 & 5 & 3 & 10b_1 \\ 0 & 0 & 1 & -1 & 10b_1 - 10b_2 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 - 5R_2} \left(\begin{array}{cccc|c} 1 & 1 & 0 & 8 & -40b_1 + 50b_2 \\ 0 & 0 & 1 & -1 & 10b_1 - 10b_2 \\ 0 & 0 & 0 & 0 & b_2 + b_3 - 2b_1 \end{array}\right)$$

c) The last row in the row-reduced matrix translates into $0 = b_2 + b_3 - 2b_1$. Hence the system of equations is inconsistent unless $\boxed{b_2 + b_3 - 2b_1 = 0}$.

d) This is the plane in $\mathbf{R}^3$ given by $-2b_1 + b_2 + b_3 = 0$.

e) Rearranging, this is the set of points (b_1, b_2, b_3) where $b_1 = \frac{1}{2}(b_2 + b_3)$, i.e., where b_1 is the average of b_2 and b_3. Hence it is impossible for b_1 to be larger than both b_2 and b_3.

You should argue for the second grading scheme if your final grade was higher than your midterm grade; otherwise you should argue for the third.

2. For each of the following, give an example if it is possible. If it is not possible, justify why there is no such example.

a) A 3×4 matrix A in RREF with 2 pivot columns, so that for some vector b, the system $Ax = b$ has exactly three free variables.

b) A homogeneous linear system with no solution.

c) A 5×3 matrix in RREF such that $Ax = 0$ has a non-trivial solution.

d) A consistent system $Ax = b$ whose solution set is a span.

Solution.

a) Not possible. If A had 2 pivot columns and 3 free variables then it would have 5 columns.

b) Not possible. Any homogeneous linear system has the trivial solution.

c) Yes. For the matrix A below, the system $Ax = 0$ will have two free variables and thus infinitely many solutions.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

d) Yes. Take any homogeneous matrix equation $Ax = 0$, then the solution set will be a span.

3. Suppose the solution set of a certain system of linear equations is given by

$$x_1 = 9 + 8x_4, \qquad x_2 = -9 - 14x_4, \qquad x_3 = 1 + 2x_4, \qquad x_4 = x_4 \ \ (x_4 \text{ free}).$$

Write the solution set in parametric vector form. Describe the set geometrically.

Solution.

In parametric vector form, the solutions are given by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 + 8x_4 \\ -9 - 14x_4 \\ 1 + 2x_4 \\ x_4 \end{pmatrix} = \begin{pmatrix} 9 \\ -9 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 8 \\ -14 \\ 2 \\ 1 \end{pmatrix}.$$

This is the line in $\mathbf{R}^4$ through $\begin{pmatrix} 9 \\ -9 \\ 1 \\ 0 \end{pmatrix}$ parallel to Span $\left\{ \begin{pmatrix} 8 \\ -14 \\ 2 \\ 1 \end{pmatrix} \right\}$.

4. **a)** What best describes the span of the columns of $\begin{pmatrix} 0 & 2 & 0 \\ 1 & 3 & 1 \\ 1 & 1 & 0 \end{pmatrix}$? Justify your answer.

(I) It is a plane through the origin.

(II) It is three lines through the origin.

(III) It is all of $\mathbf{R}^3$.

(IV) It is a plane, plus the line through the origin and the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

b) Let $A = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}$. Does $Ax = b$ have at least one solution for each b in $\mathbf{R}^3$? If yes, justify your answer. If not, write a vector in $\mathbf{R}^3$ so that $Ax = b$ is inconsistent.

Solution.

a) It is all of $\mathbf{R}^3$. If we row-reduce the matrix, we will see it has a pivot in every row, so its columns span $\mathbf{R}^3$.

b) No. The first and third columns of A are scalar multiples of each other, so we can see the three vectors cannot span $\mathbf{R}^3$. Note that any vector in the column span of A has first coordinate equal to the negative of the third coordinate, so (for example) $Ax = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ will be inconsistent.

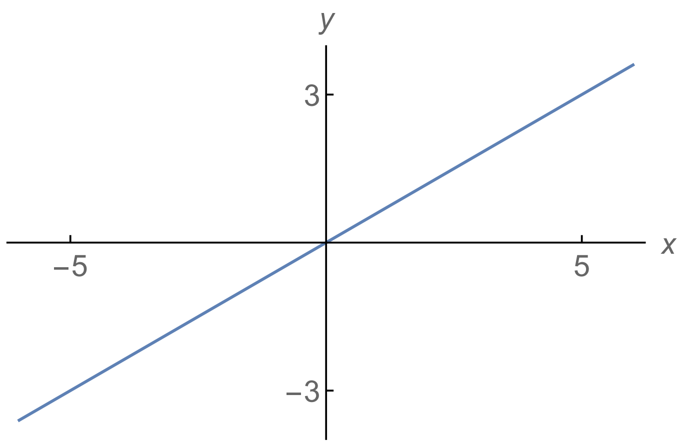
5. Let $A = \begin{pmatrix} 5 & -5 & 10 \\ 3 & -3 & 6 \end{pmatrix}$. Draw the column span of A.

Solution.

Let $v_1,\ v_2,\ v_3$ be the columns of A. The columns are scalar multiples of each other: $v_2 = -v_1$ and $v_3 = 2v_1$. This means that all three vectors are on the same line through the origin, so

$$\mathrm{Span}\{v_1, v_2, v_3\} = \mathrm{Span}\{v_1\} = \mathrm{Span}\left\{\begin{pmatrix} 5 \\ 3 \end{pmatrix}\right\}.$$

This is the line through the origin and $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$, namely the line $y = \dfrac{3x}{5}$.



6. Consider the following consistent system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 &= -2 \\ 3x_1 + 4x_2 + 5x_3 + 6x_4 &= -2 \\ 5x_1 + 6x_2 + 7x_3 + 8x_4 &= -2 \end{aligned}$$

a) Find the parametric vector form for the general solution.

b) Find the parametric vector form of the corresponding *homogeneous* equations.

Solution.

a) We put the equations into an augmented matrix and row reduce:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 3 & 4 & 5 & 6 & -2 \\ 5 & 6 & 7 & 8 & -2 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & -2 & -4 & -6 & 4 \\ 0 & -4 & -8 & -12 & 8 \end{array}\right) \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & -2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccc|c} 1 & 0 & -1 & -2 & 2 \\ 0 & 1 & 2 & 3 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

This means x_3 and x_4 are free, and the general solution is

$$\begin{cases} x_1 \phantom{{}+x_2} - x_3 - 2x_4 = 2 \\ \phantom{x_1 +{}} x_2 + 2x_3 + 3x_4 = -2 \end{cases} \implies \begin{cases} x_1 = x_3 + 2x_4 + 2 \\ x_2 = -2x_3 - 3x_4 - 2 \\ x_3 = x_3 \\ x_4 = x_4 \end{cases}$$

This gives the parametric vector form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 2 \\ -3 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix}.$$

b) Part (a) shows that the solution set of the original equations is the translate of

$$\operatorname{Span}\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}\right\} \quad \text{by} \quad \begin{pmatrix}2\\-2\\0\\0\end{pmatrix}.$$

We know that the solution set of the homogeneous equations is the parallel plane through the origin, so it is

$$\operatorname{Span}\left\{\begin{pmatrix}1\\-2\\1\\0\end{pmatrix},\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}\right\}.$$

Hence the parametric vector form is

$$\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix} = x_3\begin{pmatrix}1\\-2\\1\\0\end{pmatrix} + x_4\begin{pmatrix}2\\-3\\0\\1\end{pmatrix}.$$

7. Which of the following must be true for any set of seven vectors in $\mathbf{R}^5$? Answer "yes", "no", or "maybe" in each case.

a) The vectors span $\mathbf{R}^5$.

b) If we put the seven vectors as the columns of a matrix A, then the matrix equation $Ax = 0$ must have infinitely many solutions.

c) Suppose we put the seven vectors as the columns of a matrix A. Then for each b in $\mathbf{R}^5$, the matrix equation $Ax = b$ must be consistent.

d) If every vector b in $\mathbf{R}^5$ can be written as a linear combination of our seven vectors, then in fact every b in $\mathbf{R}^5$ can be written in *infinitely many* different ways as a linear combination of our seven vectors.

Solution.

a) Maybe.

b) Yes.

c) Maybe.

d) Yes. By assumption, the matrix A whose columns are our seven vectors has $\mathbf{R}^5$ as its column span, so A will have a pivot in every row. Therefore, A will have 5 pivot columns, so it will have 2 columns without pivots. This means that $Ax = b$ will be consistent no matter what b is, and there will be two free variables.