Math 1553 Worksheet §§3.5-4.3

Solutions

1. True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

a) If A and B are $n \times n$ matrices and both are invertible, then the inverse of AB is $A^{-1}B^{-1}$.

b) If A is a real 3×3 matrix, then A cannot satisfy $A^2 = -I$.

c) Suppose A is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of A. Then A must be invertible.

d) If $\det(A) = 1$ and c is a scalar, then $\det(cA) = c\det(A)$.

Solution.

a) False. $(AB)^{-1} = B^{-1}A^{-1}$.

b) True. If $A^2 = -I_3$ then

$$\det(A^2) = \det(-I), \qquad (\det\ A)^2 = \det\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} = -1,$$

which is impossible because the square of a real number can never be negative.

Note that it is possible to have $A^2 = -I_2$ if A is 2×2. For example, if A represents rotation counterclockwise by 90 degrees, then A^2 does rotation counterclockwise by 180 degrees which is given by $-I_2$.

c) True. If the columns of A span $\mathbf{R}^n$, then A is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of A span $\mathbf{R}^n$, then A has n pivots, so A has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, A is invertible.

d) False. By the properties of the determinant, scaling one row by c multiplies the determinant by c. When we take cA for an $n \times n$ matrix A, we are multiplying *each* row by c. This multiplies the determinant by c a total of n times. Thus, if A is $n \times n$ and $\det(A) = 1$, then

$$\det(cA) = c^n \det(A) = c^n(1) = c^n.$$

2. Let $A = \begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$

a) Compute $\det(A)$.

b) Compute $\det(A^{-1})$ without doing any more work.

c) Compute $\det((A^T)^5)$ without doing any more work.

d) Find the volume of the parallelepiped formed by the columns of A.

Solution.

a) The second column has three zeros, so we expand by cofactors:

$$\det\begin{pmatrix} 7 & 1 & 4 & 1 \\ -1 & 0 & 0 & 6 \\ 9 & 0 & 2 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} = -\det\begin{pmatrix} -1 & 0 & 6 \\ 9 & 2 & 3 \\ 0 & 0 & -1 \end{pmatrix}$$

Now we expand the second column by cofactors again:

$$\cdots = -2\det\begin{pmatrix} -1 & 6 \\ 0 & -1 \end{pmatrix} = (-2)(-1)(-1) = -2.$$

b) From our notes, we know $\det(A^{-1}) = \dfrac{1}{\det(A)} = -\dfrac{1}{2}$.

c) $\det(A^T) = \det(A) = -2$. By the multiplicative property of determinants, if B is any $n \times n$ matrix, then $\det(B^n) = (\det B)^n$, so

$$\det((A^T)^5) = (\det A^T)^5 = (-2)^5 = -32.$$

d) Volume of the parallelepiped is $|\det(A)| = 2$

3. Suppose we have

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 5.$$

Compute

$$\det\begin{pmatrix} d-3a & e-3b & f-3c \\ a & b & c \\ 2g & 2h & 2i \end{pmatrix}.$$

Solution.

a) Recall that the only operations that affect determinants are "row-swaps" and "row scaling". "Row replacement (through row operation)" does not change the determinant. We have the following row operations:

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} d & e & f \\ a & b & c \\ g & h & i \end{pmatrix} \xrightarrow{R_1 = R_1 - 3R_2} \begin{pmatrix} d-3a & e-3b & f-3c \\ a & b & c \\ g & h & i \end{pmatrix}$$

$$\xrightarrow{R_3 = 2R_3} \begin{pmatrix} d-3a & e-3b & f-3c \\ a & b & c \\ 2g & 2h & 2i \end{pmatrix}.$$

Therefore, we have one "row swap" which scales our determinant by -1 and we have a "row multiplied by a scalar" which scales our determinant by the scalar, which in this case is 2. Therefore, we have that

$$\det\begin{pmatrix} d-3a & e-3b & f-3c \\ a & b & c \\ 2g & 2h & 2i \end{pmatrix} = (5)(-1)(2) = -10.$$