**Math 1553 Worksheet §3.4-3.6**

Solutions

**1.** True or false. Answer true if the statement is *always* true. Otherwise, answer false. If your answer is false, either give an example that shows it is false or (in the case of an incorrect formula) state the correct formula.

**a)** If $A$ is an $n \times n$ matrix and the equation $Ax = b$ has at least one solution for each $b$ in $\mathbf{R}^n$, then the solution is *unique* for each $b$ in $\mathbf{R}^n$.

**b)** If $A$ is a $3 \times 4$ matrix and $B$ is a $4 \times 2$ matrix, then the linear transformation $Z$ defined by $Z(x) = ABx$ has domain $\mathbf{R}^3$ and codomain $\mathbf{R}^2$.

**c)** Suppose $A$ is an $n \times n$ matrix and every vector in $\mathbf{R}^n$ can be written as a linear combination of the columns of $A$. Then $A$ must be invertible.

## Solution.

**a)** True. The first part says the transformation $T(x) = Ax$ is onto. Since $A$ is $n \times n$, then it has $n$ pivots. This is the same as saying $A$ is invertible, and there is no free variable. Therefore, the equation $Ax = b$ has exactly one solution for each $b$ in $\mathbf{R}^n$.

**b)** False. In order for $Bx$ to make sense, $x$ must be in $\mathbf{R}^2$, and so $Bx$ is in $\mathbf{R}^4$ and $A(Bx)$ is in $\mathbf{R}^3$. Therefore, the domain of $Z$ is $\mathbf{R}^2$ and the codomain of $Z$ is $\mathbf{R}^3$.

**c)** True. If the columns of $A$ span $\mathbf{R}^n$, then $A$ is invertible by the Invertible Matrix Theorem. We can also see this directly without quoting the IMT:

If the columns of $A$ span $\mathbf{R}^n$, then $A$ has $n$ pivots, so $A$ has a pivot in each row and column, hence its matrix transformation $T(x) = Ax$ is one-to-one and onto and thus invertible. Therefore, $A$ is invertible.

**2.** $A$ is $m \times n$ matrix, $B$ is $n \times m$ matrix. Select all correct answers from the box. It is possible to have more than one correct answer.

**a)** Suppose $x$ is in $\mathbf{R}^m$. Then $ABx$ *must be* in:
$\boxed{\mathrm{Col}(A),\quad \mathrm{Nul}(A),\quad \mathrm{Col}(B),\quad \mathrm{Nul}(B)}$

**b)** Suppose $x$ in $\mathbf{R}^n$. Then $BAx$ *must be* in:
$\boxed{\mathrm{Col}(A),\quad \mathrm{Nul}(A),\quad \mathrm{Col}(B),\quad \mathrm{Nul}(B)}$

**c)** If $m > n$, then columns of $AB$ could be linearly $\boxed{independent, \quad dependent}$

**d)** If $m > n$, then columns of $BA$ could be linearly $\boxed{independent, \quad dependent}$

**e)** If $m > n$ and $Ax = 0$ has nontrivial solutions, then columns of $BA$ could be linearly $\boxed{independent, \quad dependent}$

## Solution.

Recall, $AB$ can be computed as $A$ multiplying every column of $B$. That is $AB = \begin{pmatrix} Ab_1 & Ab_2 & \cdots Ab_m \end{pmatrix}$ where $B = \begin{pmatrix} b_1 & b_2 & \cdots b_m \end{pmatrix}$.

**a)** $\boxed{\mathrm{Col}(A)}$. Denote $w := Bx$, which is a vector in $\mathbf{R}^n$. $ABx = A(Bx)$ is multiplying $A$ with $w$ which will end up with "linear combination of columns of $A$". So $ABx$ is in Col($A$).

**b)** $\boxed{\mathrm{Col}(B)}$. Similarly, $BAx = B(Ax)$ is multiplying $B$ with $Ax$, a vector in $R^m$, which will end up with "linear combination of columns of $B$". So $BAx$ is in Col($B$).

**c)** $\boxed{dependent}$. Since $m > n$ means $A$ matrix can have at most $n$ pivots. So $dim(\mathrm{Col}(A)) \leq n$. Notice from first question we know $\mathrm{Col}(AB) \subset \mathrm{Col}(A)$ which has dimension at most $n$. That means $AB$ can have at most $n$ pivots. But $AB$ is $m \times m$ matrix, then columns of $AB$ must be dependent.

**d)** $\boxed{independent, dependent}$. Both are possible. Since $m > n$ means $B$ matrix can have at most $n$ pivots. then $\mathrm{Col}(BA) \subset \mathrm{Col}(B)$ means $BA$ can have at most $n$ pivots. Since $BA$ is $n \times n$ matrix, then the columns of $BA$ will be linearly independent when there are $n$ pivots or linearly dependent when there are less than $n$ pivots. Here are two examples.

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \text{then } BA = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

**e)** $\boxed{dependent}$. From the second example above, $BA$ has dependent columns, we know "dependent" is one possible answer. Now to see if "independent" is also possible, we need to find out if $BA$ could have $n$ pivots.

Since $Ax = 0$ has nontrivial solution say $x^*$, then $x^*$ is also a nontrivial solution of $BAx = 0$. That means $BA$ has free variables, and it can not have $n$ pivots. So columns of $BA$ must be linearly dependent.

To summarize what we are actually study here, there are several relations between these subspaces. The symbol $\subseteq$ means “is contained in (or possibly equal to)..."

$$\mathrm{Col}(AB) \subseteq \mathrm{Col}(A);$$
$$\mathrm{Col}(BA) \subseteq \mathrm{Col}(B);$$
$$\mathrm{Nul}(A) \subseteq \mathrm{Nul}(BA);$$
$$\mathrm{Nul}(B) \subseteq \mathrm{Nul}(AB);$$

**3.** Consider the following linear transformations:

$T : \mathbf{R}^3 \longrightarrow \mathbf{R}^2$   $T$ projects onto the $xy$-plane, forgetting the $z$-coordinate

$U : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$   $U$ rotates clockwise by 90°

$V : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$   $V$ scales the $x$-direction by a factor of 2.

Let $A, B, C$ be the matrices for $T, U, V$, respectively.

**a)** Write $A$, $B$, and $C$.

**b)** Compute the matrix for $U \circ V \circ T$.

**c)** Describe $U^{-1}$ and $V^{-1}$, and compute their matrices.

## Solution.

**a)** We plug in the unit coordinate vectors:

$$T(e_1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad T(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad T(e_3) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$U(e_1) = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \quad U(e_2) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \Longrightarrow \quad B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} .$$

$$V(e_1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix} \quad V(e_2) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \Longrightarrow \quad C = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

**b)** $BCA = \begin{pmatrix} 0 & 1 & 0 \\ -2 & 0 & 0 \end{pmatrix}$.

**c)** Intuitively, if we wish to "undo" $U$, we can imagine that $\begin{pmatrix} x \\ y \end{pmatrix}$. To do this, we need to rotate it 90° *counterclockwise*. Therefore, $U^{-1}$ is counterclockwise rotation by 90°.

Similarly, to undo the transformation $V$ that scales the $x$-direction by 2, we need to scale the $x$-direction by 1/2, so $V^{-1}$ scales the $x$-direction by a factor of 1/2.

Their matrices are, respectively,

$$B^{-1} = \frac{1}{0 \cdot 0 - (-1) \cdot 1} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and

$$C^{-1} = \frac{1}{2 \cdot 1 - 0 \cdot 0} \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$$