Math 1553 Worksheet §2.3, S2.4

Solutions

1. True or false. If the statement is *always* true, answer True. Otherwise, answer False.

a) Suppose A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$. If $Ax = b$ is inconsistent, then A does not have a pivot in every column.

b) If A is a 4×3 matrix, then the equation $Ax = b$ is inconsistent for some b in $\mathbf{R}^4$.

c) Suppose A is a 3×3 matrix with two pivots, and suppose that b is a vector so that $Ax = b$ is consistent. Then the solution set for $Ax = b$ is a plane.

Solution.

a) False, for instance consider

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

This is an inconsistent system even though A has a pivot in each column.

b) True. Any 4×3 matrix A will have at most 3 pivots, so A cannot have a pivot in every row. For example, consider the augmented matrix $\left(A \mid b\right)$ below.

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

c) False. The matrix A has two pivots, which means that the *column span* of A is plane, but this is not what the question is asking! It is asking about the solution set to $Ax = b$, not the column span of A.

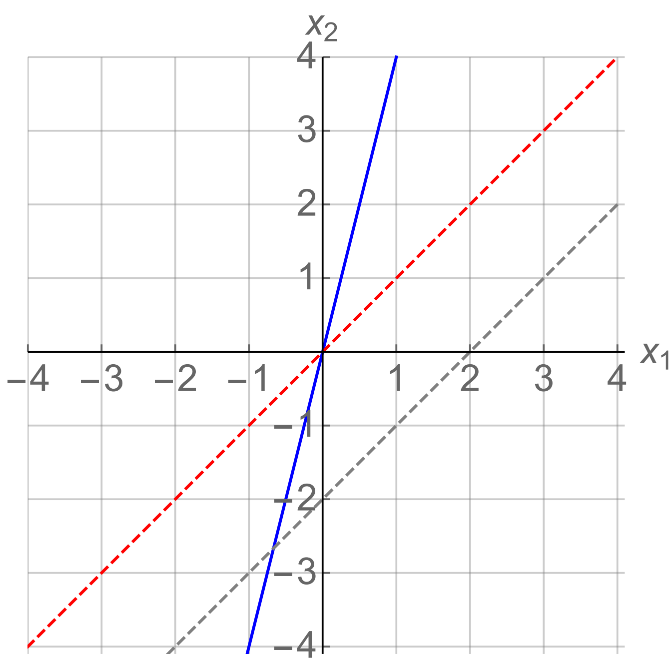
Since $Ax = b$ corresponds to a system of 3 equations in 3 variables, the fact that A has two pivots means that the system will have exactly one free variable, so the solution set will be a line in $\mathbf{R}^3$.

2. Let $A = \begin{pmatrix} 1 & -1 \\ 4 & -4 \end{pmatrix}$. On the same graph, draw each of the following:
(a) The span of the columns of A.

(b) The set of solutions to $Ax = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

(c) The set of solutions to $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$.
Label each of these clearly.

Solution.



The blue line is the span of columns of A: Span $\left\{\begin{pmatrix} 1 \\ 4 \end{pmatrix}\right\}$. If you draw the two column vectors, you will see they both fall on the line $x_2 = 4x_1$.

The red dashed line is the span of solutions of $Ax = 0$: Span $\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}$. To see this is the case, you can row reduce the augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right)$. That implies the solution set is the line $x_2 = x_1$.

The gray dashed line is the set of solutions to $Ax = \begin{pmatrix} 2 \\ 8 \end{pmatrix}$. To see this is the case, you can row reduce the corresponding augmented matrix to RREF, which is $\left(\begin{array}{cc|c} 1 & -1 & 2 \\ 0 & 0 & 0 \end{array}\right)$. That implies the solution set is the line $x_1 = 2 + x_2$ (where x_2 is free) which yields

parametric form
$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 + x_2 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$
In other words, this solution set is the line through $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ parallel to the span of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

3. Find the set of solutions to $x_1 - 3x_2 + 5x_3 = 0$ and write your answer in parametric vector form. Next, find the set of solutions to $x_1 - 3x_2 + 5x_3 = 3$ and write the solutions in parametric vector form. How do the solution sets compare geometrically?

Solution.

The homogeneous system $x_1 - 3x_2 + 5x_3 = 0$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 0 \end{array}\right)$, which has two free variables x_2 and x_3.

$$x_1 = 3x_2 - 5x_3 \qquad x_2 = x_2 \text{ (free)} \qquad x_3 = x_3 \text{ (free)}.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

The solution set for $x_1 - 3x_2 + 5x_3 = 0$ is the plane spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.

The nonhomogeneous system $x_1 - 3x_2 + 5x_3 = 3$ corresponds to the augmented matrix $\left(\begin{array}{ccc|c} 1 & -3 & 5 & 3 \end{array}\right)$ which has two free variables x_2 and x_3.

$$x_1 = 3 + 3x_2 - 5x_3 \qquad x_2 = x_2 \qquad x_3 = x_3.$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 + 3x_2 - 5x_3 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 3x_2 \\ x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -5x_3 \\ 0 \\ x_3 \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}}.$$

This solution set (red) is the *translation by* $\begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$ of the plane (green) spanned by $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$.