Math 1553 Worksheet §2.1, §2.2

Solutions

1. Consider the system of linear equations

$$\begin{aligned} x + 2y &= 7 \\ 2x + y &= -2 \\ -x - y &= 4. \end{aligned}$$

Question: What are the solutions (if there are any) to the system?

a) Formulate this question as a question about an augmented matrix.

b) Answer the question using row reduction.

c) Formulate this question as a vector equation.

d) What does this question mean in terms of spans?

e) Answer part (d) using the interactive demo.

Solution.

a) What are the solutions (if there are any) for the corresponding augmented matrix below?

$$\left(\begin{array}{cc|c} 1 & 2 & 7 \\ 2 & 1 & -2 \\ -1 & -1 & 4 \end{array}\right)$$

b) Row reducing the matrix in part **a)** yields

$$\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right),$$

so there are no solutions to the system of linear equations.

c) What are the solutions (if there are any) to the following vector equation?

$$x\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + y\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$$

d) Is $\begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$ in Span $\left\{\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}\right\}$?

e) The picture in the interactive demo shows that $\begin{pmatrix} 7 \\ -2 \\ 4 \end{pmatrix}$ is not in the span of $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$, which is the same as saying that the corresponding system of linear equations is inconsistent.

2. **a)** Write a set of three different vectors whose span is a line in $\mathbf{R}^3$.

b) Write a set of three different vectors whose span is a plane in $\mathbf{R}^3$.

c) Write a set of three vectors whose span is only a single point in $\mathbf{R}^3$.

d) In each of the above questions, if you form the matrix A whose columns are the three vectors, how many pivots does A have?

Solution.

a) Just choose any vector that spans your favorite line, then pick the other vectors to be within that line. For example, choose $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\0\\0\end{pmatrix}$, and $v_3 = \begin{pmatrix}3\\0\\0\end{pmatrix}$, which span the x-axis within $\mathbf{R}^3$.

b) Similar to above. Just choose any two vectors that span your favorite plane, then pick your third vector to be within that plane. For example, choose $v_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}$, and $v_3 = \begin{pmatrix}1\\1\\0\end{pmatrix}$. The span of these three vectors is the xy-plane in $\mathbf{R}^3$.

c) The span of any three vectors v_1, v_2, v_3 in $\mathbf{R}^3$ must contain the origin, since

$$0v_1 + 0v_2 + 0v_3 \text{ is automatically the zero vector.}$$

There is only one possibility for this answer: we must choose $v_1 = v_2 = v_3 = \begin{pmatrix}0\\0\\0\end{pmatrix}$. If our list had contained a nonzero vector, then the span would include that nonzero vector and all scalar multiples of it (including the zero vector).

d) For a) the matrix A has one pivot. For b) the matrix has two pivots. For c) the matrix has no pivots.

3. Jameson Locke has challenged you to find a hidden treasure, located at some point (a, b, c). He has honestly guaranteed you that the treasure can be found by starting at the origin and taking steps using

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \qquad v_2 = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \qquad v_3 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

By decoding the message, you have discovered that the first and second coordinates of the treasure's location are (in order) −4 and 3.

a) What is the treasure's full location?

b) Give instructions for how to find the treasure by only using v_1, v_2, and v_3. Can you do the same to get the treasure by just using v_1 and v_2?

Solution.

a) We translate this problem into linear algebra. Let c be the final entry of the treasure's location. Since Jameson has assured us that we can find the treasure using the three vectors we have been given, our problem is to find c so that $\begin{pmatrix} -4 \\ 3 \\ c \end{pmatrix}$ is a linear combination of v_1, v_2, and v_3 (in other words, find c so that the treasure's location is in in $\text{Span}\{v_1, v_2, v_3\}$). We form an augmented matrix and find when it gives a consistent system.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & c \end{array}\right) \xrightarrow[R_3=R_3+2R_1]{R_2=R_2+R_1} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & c-8 \end{array}\right) \xrightarrow{R_3=R_3-3R_2} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & c-5 \end{array}\right).$$

This system will be consistent if and only if the right column is not a pivot column, so we need $c - 5 = 0$, or $c = 5$.

The location of the treasure is $(-4, 3, 5)$.

b) Getting to the point $(-4, 3, 5)$ using the vectors v_1, v_2, and v_3 is equivalent to finding scalars x_1, x_2, and x_3 so that

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

We can rewrite this as

$$\begin{aligned} x_1 + 5x_2 - 3x_3 &= -4 \\ -x_1 - 4x_2 + x_3 &= 3 \\ -2x_1 - 7x_2 &= 5. \end{aligned}$$

We put the matrix from part (a) into RREF.

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{R_1=R_1-5R_2} \left(\begin{array}{ccc|c} 1 & 0 & 7 & 1 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

Note x_3 is the only free variable, so:

$$x_1 = 1 - 7x_3, \qquad x_2 = -1 + 2x_3 \qquad x_3 = x_3 \quad (x_3 \text{ real}).$$

Since the system has infinitely many solutions, there are infinitely many ways to get to the treasure. In fact, we can get to the treasure using v_1 and v_2 alone if we wish! If we choose the path corresponding to $x_3 = 0$, then $x_1 = 1$ and $x_2 = -1$, which means that we move 1 unit in the direction of v_1 and -1 unit in the direction of v_2. In equations:

$$\begin{pmatrix} -4 \\ 3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} - \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} + 0 \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$