Math 1553 Worksheet §5.4, 5.5

1. True or false. If the statement is always true, answer true and justify why it is true. Otherwise, answer false and give an example that shows it is false. If not explicitly stated, assume A is an $n \times n$ matrix.

a) A 3×3 matrix A can have a non-real complex eigenvalue with multiplicity 2.

b) If the RREF of A is diagonalizable, then A must be diagonalizable.

Solution.

a) No. If c is a (non-real) complex eigenvalue with multiplicity 2, then its conjugate $\bar{c}$ is an eigenvalue with multiplicity 2 since complex eigenvalues always occur in conjugate pairs. This would mean A has a characteristic polynomial of degree 4 or more, which is impossible since A is 3×3.

b) No, for example, $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable but its RREF is the identity matrix which is diagonalizable.

2. Suppose A is a 2×2 matrix satisfying

$$A\begin{pmatrix}-1\\1\end{pmatrix} = \begin{pmatrix}2\\-2\end{pmatrix}, \qquad A\begin{pmatrix}-2\\3\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}.$$

a) Diagonalize A by finding 2×2 matrices C and D (with D diagonal) so that $A = CDC^{-1}$.

b) Find A^{17}.

Solution.

a) From the information given, $\lambda_1 = -2$ is an eigenvalue for A with corresponding eigenvector $\begin{pmatrix}-1\\1\end{pmatrix}$, and $\lambda_2 = 0$ is an eigenvalue with eigenvector $\begin{pmatrix}-2\\3\end{pmatrix}$.

By the Diagonalization Theorem, $A = CDC^{-1}$ where

$$C = \begin{pmatrix}-1 & -2\\1 & 3\end{pmatrix}, \qquad D = \begin{pmatrix}-2 & 0\\0 & 0\end{pmatrix}.$$

b) We find $C^{-1} = \dfrac{1}{-3+2}\begin{pmatrix}3 & 2\\-1 & -1\end{pmatrix} = \begin{pmatrix}-3 & -2\\1 & 1\end{pmatrix}.$

$$\begin{aligned}A^{17} = CD^{17}C^{-1} &= \begin{pmatrix}-1 & -2\\1 & 3\end{pmatrix}\begin{pmatrix}(-2)^{17} & 0\\0 & 0\end{pmatrix}\begin{pmatrix}-3 & -2\\1 & 1\end{pmatrix}\\ &= \begin{pmatrix}-1 & -2\\1 & 3\end{pmatrix}\begin{pmatrix}3\cdot 2^{17} & 2^{18}\\0 & 0\end{pmatrix}\\ &= \begin{pmatrix}-3\cdot 2^{17} & -2^{18}\\3\cdot 2^{17} & 2^{18}\end{pmatrix}.\end{aligned}$$

3. Let $A = \begin{pmatrix}2 & 3\\-1 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}2 & 3\\-1 & 1\end{pmatrix}^{-1}$, and let $x = \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix}$. What happens to $A^n x$ as n gets very large?

Solution.

We are given diagonalization of A, which gives us the eigenvalues and eigenvectors.

$$\begin{aligned}A^n x = A^n\left(\begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}3\\1\end{pmatrix}\right) &= A^n\begin{pmatrix}2\\-1\end{pmatrix} + A^n\begin{pmatrix}3\\1\end{pmatrix}\\ &= 1^n\begin{pmatrix}2\\-1\end{pmatrix} + \left(\frac{1}{2}\right)^n\begin{pmatrix}3\\1\end{pmatrix}\\ &= \begin{pmatrix}2\\-1\end{pmatrix} + \begin{pmatrix}\frac{3}{2^n}\\\frac{1}{2^n}\end{pmatrix}.\end{aligned}$$

As n gets very large, the entries in the second vector above approach zero, so $A^n x$ approaches $\begin{pmatrix}2\\-1\end{pmatrix}$. For example, for $n = 15$,

$$A^{15}x \approx \begin{pmatrix}2.00009\\-0.999969\end{pmatrix}.$$

4. Let $A = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}$. Find all eigenvalues of A. For each eigenvalue, find an associated eigenvector.

Solution.

The characteristic polynomial is

$$\lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 2\lambda + 5$$

$$\lambda^2 - 2\lambda + 5 = 0 \iff \lambda = \frac{2 \pm \sqrt{4-20}}{2} = \frac{2 \pm 4i}{2} = 1 \pm 2i.$$

For the eigenvalue $\lambda = 1-2i$, we use the shortcut trick you may have seen in class: the first row $\begin{pmatrix} a & b \end{pmatrix}$ of $A - \lambda I$ will lead to an eigenvector $\begin{pmatrix} -b \\ a \end{pmatrix}$ (or equivalently, $\begin{pmatrix} b \\ -a \end{pmatrix}$ if you prefer).

$$\left(A - (1-2i)I \mid 0\right) = \left(\begin{array}{cc|c} 2i & 2 & 0 \\ (*) & (*) & 0 \end{array}\right) \implies v = \begin{pmatrix} -2 \\ 2i \end{pmatrix}.$$

From the correspondence between conjugate eigenvalues and their eigenvectors, we know (without doing any additional work!) that for the eigenvalue $\lambda = 1+2i$, a corresponding eigenvector is $w = \overline{v} = \begin{pmatrix} -2 \\ -2i \end{pmatrix}$.

If you used row-reduction for finding eigenvectors, you would find $v = \begin{pmatrix} i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1-2i$, and $w = \begin{pmatrix} -i \\ 1 \end{pmatrix}$ as an eigenvector for eigenvalue $1+2i$.