Math 1553 Worksheet: Fundamentals, §1.1, and beginning §1.2

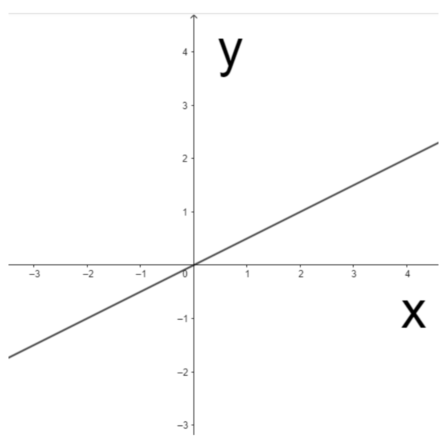
Solutions

1. **a)** (Warm-up) Draw the set of all points in $\mathbf{R}^2$ that satisfy the equation $x-2y=0$, where we use (x,y) to denote points in $\mathbf{R}^2$.

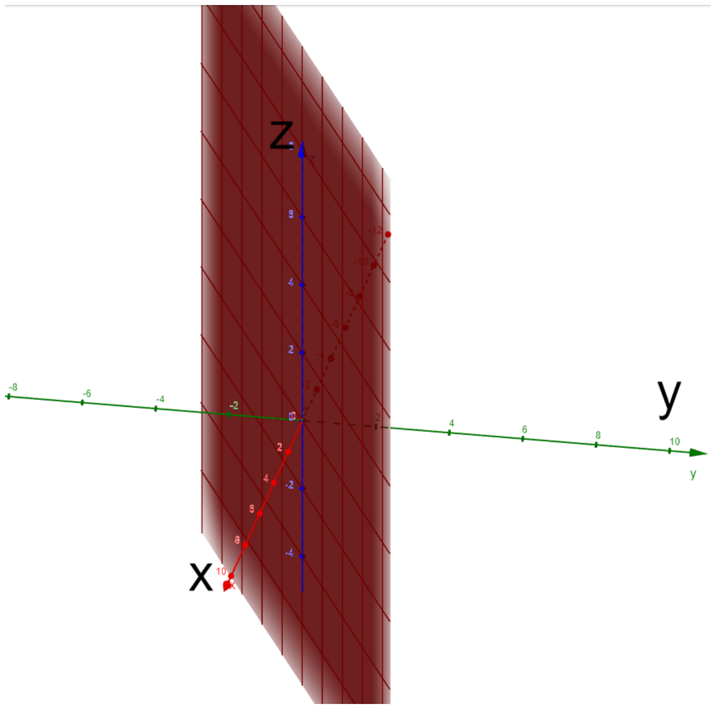
b) Draw the set of all points in $\mathbf{R}^3$ that satisfy the equation $x-2y=0$, where we use (x,y,z) to denote points in $\mathbf{R}^3$. Geometrically, does this set form a line, a plane, or something else?

Solution.

a) This is the line $y=x/2$.



b) *Adding $2y$ to both sides, and dividing, gives $y=x/2$* like in part (a). However, points in $\mathbf{R}^3$ have three coordinates (x,y,z) rather than just two coordinates, so we will get a *plane* rather than a line: *$y=x/2$* but z can be anything we want. In other words, the points in $\mathbf{R}^3$ satisfying *$x-2y=0$* are the points *$(x,x/2,z)$* where x and z are any real numbers.



2. Richard Straker has *seven* light switches in order along a wall. He records which lights are on and which lights are off. To save time, he uses 0 to represent "off" and using 1 to represent "on" for each light.

a) Write an element of $\mathbf{R}^n$ (for some n) that represents the situation when *the last three lights are on, and the first four are off*. What is n?

b) Repeat part (a) when *the first three lights are on and the rest are off*.

Solution.

a) If *the last three lights are on*, then each gets a 1 and the rest get a 0, so we can represent this by $(0,0,0,0,1,1,1)$, which is in $\mathbf{R}^7$ because it has *seven* coordinates (one for each light).

b) If *the first three lights are on*, then each gets a 1 and the rest get a 0, so we can represent this by $(1,1,1,0,0,0,0)$, which is in $\mathbf{R}^7$.

3. Find all values of h so that the lines $x+hy=-5$ and $2x-8y=6$ do *not* intersect, and indicate what this means for the set of solutions to the linear system of equations

$$x+hy=-5$$
$$2x-8y=6.$$

For all h so that the lines do not intersect, draw the line $x+hy=-5$ and the line $2x-8y=6$ to verify that they do not intersect.

Solution.

The lines fail to intersect precisely when the corresponding system of linear equations is inconsistent (i.e. has no solutions).

To do this problem, we can use basic algebra, geometric intuition, or row operations.

Using basic algebra: Let's see what happens when the lines *do* intersect. In that case, there is a point (x, y) where

$$\begin{aligned} x + hy &= -5 \\ 2x - 8y &= \ \ 6. \end{aligned}$$

Subtracting twice the first equation from the second equation gives us

$$\begin{aligned} x + \qquad\quad hy &= -5 \\ (-8-2h)y &= \ 16. \end{aligned}$$

If $-8-2h=0$ (so $h=-4$), then the second line is $0 \cdot y = 16$, which is impossible. In other words, if $h=-4$ then we cannot find a solution to the system of two equations, so the two lines *do not* intersect.

On the other hand, if $h \neq -4$, then we can solve for y above:

$$(-8-2h)y = 16 \qquad y = \frac{16}{-8-2h} \qquad y = \frac{8}{-4-h}.$$

We can now substitute this value of y into the first equation to find x at the point of intersection:

$$x + hy = -5 \qquad x + h \cdot \frac{8}{-4-h} = -5 \qquad x = -5 - \frac{8h}{-4-h}.$$

Therefore, the lines fail to intersect if and only if $\boxed{h=-4}$.

Using intuition from geometry in $\mathbf{R}^2$: Two non-identical lines in $\mathbf{R}^2$ will fail to intersect, if and only if they are parallel. The second line is $y = \frac{1}{4}x - \frac{3}{4}$, so its slope is $\frac{1}{4}$.

If $h \neq 0$, then the first line is $y = -\frac{1}{h}x - \frac{5}{h}$, so the lines are parallel when $-\frac{1}{h} = \frac{1}{4}$, which means $h=-4$. In this case, the lines are $y = \frac{1}{4}x + \frac{5}{4}$ and $y = \frac{1}{4}x - \frac{3}{4}$, so they are parallel non-intersecting lines.

(If $h=0$ then the first line is vertical and the two lines intersect when $x=-5$).

Using row operations: The problem could be done using augmented matrices, which will soon become our main method for solving systems of equations.

$$\left(\begin{array}{cc|c} 1 & h & -5 \\ 2 & -8 & 6 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{cc|c} 1 & h & -5 \\ 0 & -8-2h & 16 \end{array}\right).$$

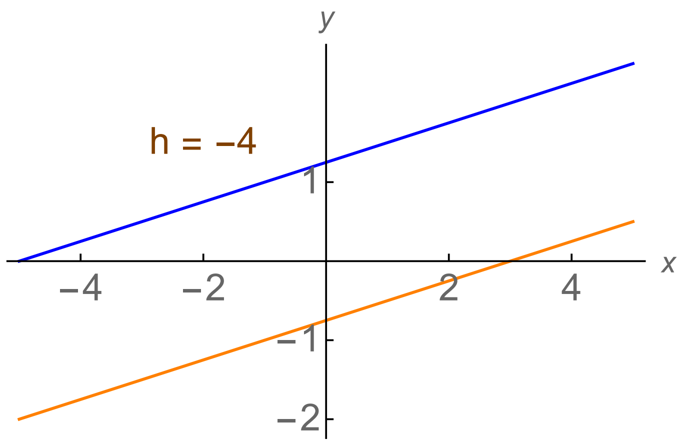
If $-8-2h=0$ (so $h=-4$), then the second equation is $0=16$, so our system has no solutions. In other words, the lines do not intersect.

If $h \neq -4$, then the second equation is $(-8-2h)y = 16$, so

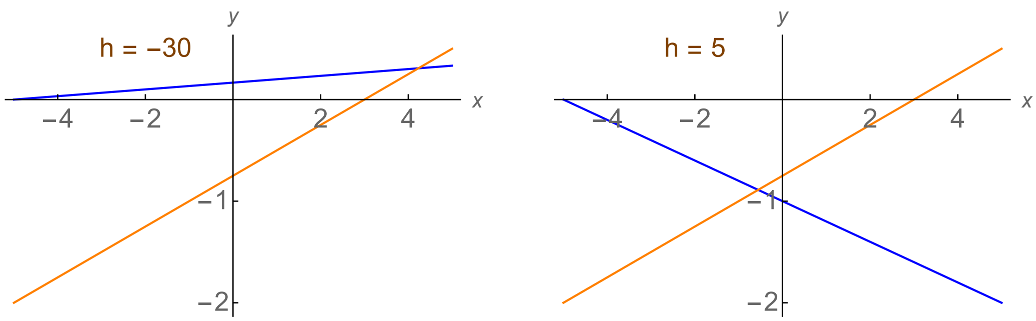
$$y = \frac{16}{-8-2h} = \frac{8}{-4-h} \qquad \text{and} \qquad x = -5 - hy = -5 - \frac{8h}{-4-h},$$

and the lines intersect at (x, y). Therefore, our answer is $h=-4$.

Here are the two lines for $h = -4$, and we can see they are different parallel lines.



If we vary h away from -4, then the blue and orange lines will have different slopes and will inevitably intersect. For example,



4. Consider the following three planes, where we use (x, y, z) to denote points in $\mathbf{R}^3$:

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ 2x + 5y + 2z &= -1 \\ y + 3z &= 8 \end{aligned}$$

Determine if all three of the planes intersect. If so, do they intersect at a single point, a line, or a plane?

Solution.

We put the system in augmented matrix form and row-reduce. Since the numbers get a bit messy in the first equation if we divide by 2, we will wait until the end to do this.

$$\left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 2 & 5 & 2 & -1 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_2 = R_2 - R_1} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 1 & 3 & 8 \end{array}\right) \xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 5 & 10 \end{array}\right)$$

$$\xrightarrow{R_3=R_3*\frac{1}{5}} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_2=R_2+2R_3} \left(\begin{array}{ccc|c} 2 & 4 & 4 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1=R_1-4R_3} \left(\begin{array}{ccc|c} 2 & 4 & 0 & -7 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

$$\xrightarrow{R_1=R_1-4R_2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & -15 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right) \xrightarrow{R_1=R_1-4R_2} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{15}{2} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array}\right).$$

Thus, we see that $x = -\frac{15}{2}$, $y = 2$, $z = 2$. In other words, the planes intersect at the point $(x, y, z) = (-\frac{15}{2}, 2, 2)$.

A method without using row reduction would be to solve everything in long-hand form. Subtracting the first equation from the second gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ y + 3z &= 8. \end{aligned}$$

Next, subtracting the second equation from the third gives us

$$\begin{aligned} 2x + 4y + 4z &= 1 \\ y - 2z &= -2 \\ 5z &= 10, \end{aligned}$$

so $z = 2$. We can back-substitute to find y and then x. The second equation is $y-2z = -2$, so $y-2(2) = -2$, thus $y = 2$. The first equation is $2x+4(2)+4(2) = 1$, so $2x = -15$, thus $x = -15/2$. We have found that the planes intersect at the point

$$\left(-\frac{15}{2},\ 2,\ 2\right).$$