Math 1553 Sample Midterm 3, Fall 2024

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

This is a practice exam. It was compiled by modifying one or two past midterms. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

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1. For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   In each statement, A is a matrix whose entries are **real numbers**.

   (a) Suppose A is a 3×3 matrix and there is some b in $\mathbf{R}^3$ so that the equation $Ax = b$ has exactly one solution. Then A must be invertible.

   ○ True

   ○ False

   (b) If A is an $n \times n$ matrix and $\det(A) = 0$, then $\lambda = 0$ must be an eigenvalue of A.

   ○ True

   ○ False

   (c) There is a 3×3 real matrix A whose eigenvalues are -1, 3, and $2 + i$.

   ○ True

   ○ False

   (d) Suppose A is a 4×4 matrix and its eigenvalues are

   $$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \lambda_3 = 5, \quad \lambda_4 = 7.$$

   Then A must be diagonalizable.

   ○ True

   ○ False

   (e) If A is a 5×5 matrix with characteristic polynomial

   $$\det(A - \lambda I) = -\lambda(\lambda + 2)(\lambda - 4)^3,$$

   then the null space of A must be a line.

   ○ True

   ○ False

2. Parts (a) through (d) are unrelated. You do not need to show your work on this page.

(a) (2 points) Let $A = \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$. Find A^{-1}. Clearly fill in the correct bubble below.

○ $\frac{1}{10}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ ○ $\frac{1}{10}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$ ○ $-\frac{1}{2}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

○ $\frac{1}{10}\begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$ ○ $\frac{1}{10}\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ ○ $-\frac{1}{2}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$

(b) (3 points) Suppose A is an invertible matrix whose inverse is given by

$$A^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(i) Suppose b is a vector in $\mathbf{R}^3$. How many solutions will the equation $Ax = b$ have? Clearly fill in the correct bubble below.

○ None ○ Exactly one ○ Infinitely many ○ Not enough info

(ii) Which **one** of the vectors below is a solution to $Ax = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$?

○ $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 0 \\ -1/3 \end{pmatrix}$

(c) (2 pts) Suppose $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det \begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}$.

Clearly fill in the correct bubble below.

○ 1 ○ -1 ○ 2 ○ -2

○ 4 ○ -4 ○ 8 ○ -8

(d) (3 points) Suppose A and B are 2×2 matrices satisfying

$$\det(A) = 6, \qquad \det(B) = -3.$$

Which of the following statements must be true? Clearly fill in the bubble for all that apply.

○ AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

○ $\det(3B^{-1}) = -1$.

○ $A - 6I$ is not invertible.

3. Parts (a), (b), (c), and (d) are unrelated. There is no work required on this page.

(a) Suppose A is an $n \times n$ matrix. Which **one** of the following statements is **not** correct?

○ An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.

○ An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.

○ An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.

○ An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

(b) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = 7x$, and let A be the standard matrix for T, so $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$.
In the blank below, write one eigenvector v in the (-1)-eigenspace of A.

$$v = \begin{pmatrix} \quad \\ \quad \end{pmatrix}$$

(c) (2 points) Let $A = \begin{pmatrix} -1 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Which **one** of the following describes the 1-eigenspace of A?

○ $\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix}\right\}$

○ $\text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}\right\}$ ○ All of $\mathbf{R}^3$

(d) (4 points) Let $A = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$. Which of the following are true? Clearly mark all that apply.

○ The eigenvalues of A are $1/2$ and 1.

○ For each vector x in $\mathbf{R}^2$, it is the case that $A^n x$ approaches $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ as n becomes large.

○ $\text{Nul}(A - I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$.

○ $A^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

4. You do not need to show your work on this problem except in part (b). Parts (a), (b), and (c) are unrelated.

   (a) (2 points) Find all values of c so that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{pmatrix} 4 & -3 \\ 4 & c \end{pmatrix}$. Clearly fill in the correct bubble below.

   ○ $c = -3$ only ○ $c = -4$ only ○ $c = 4$ only ○ $c = -6$ only

   ○ All c except -4 ○ All c except -6 ○ All c except 6

   (b) (4 points) Let $B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$ and define a matrix transformation by $T(x) = Bx$. Find the area of $T(S)$, where S is the triangle with vertices $(-1, 1)$, $(2, 3)$, and $(5, 2)$.

   (c) (4 points) Suppose A is a 3×3 matrix. Which of the following statements are true? Clearly circle all that apply.

   ○ If B is a 3×3 matrix that has the same reduced row echelon form as A, then the eigenvalues of B are the same as the eigenvalues of A.

   ○ If $\lambda = 3$ is an eigenvalue of A, then the equation $Ax = 3x$ must have infinitely many solutions.

   ○ If the equation $(A - 2I)x = 0$ has only the trivial solution, then 2 is not an eigenvalue of A.

   ○ It is impossible for A to have 4 different eigenvalues.

5. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Consider the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$.

(a) Find the characteristic polynomial of A and the eigenvalues of A.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(c) Is A diagonalizable? If yes, find an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$ and write them in the space provided below. If no, justify why A is not diagonalizable.

$$C = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

6. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

(a) Let $A = \begin{pmatrix} 5 & -5 \\ 4 & -3 \end{pmatrix}$.

(i) (4 points) Find the complex eigenvalues of A. Fully simplify your answer.

(ii) (3 points) For the eigenvalue with *positive* imaginary part, find one corresponding eigenvector v. Enter your answer in the space below.

$$v = \begin{pmatrix} \\ \\ \\ \end{pmatrix}$$

(b) (3 points) Given that

$$\det \begin{pmatrix} 0 & -1 & 3 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 28, \quad \det \begin{pmatrix} 4 & 2 & -1 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 8, \quad \text{and } \det \begin{pmatrix} 4 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 4 & 2 \end{pmatrix} = -56,$$

compute the determinant of the 4×4 matrix W below.

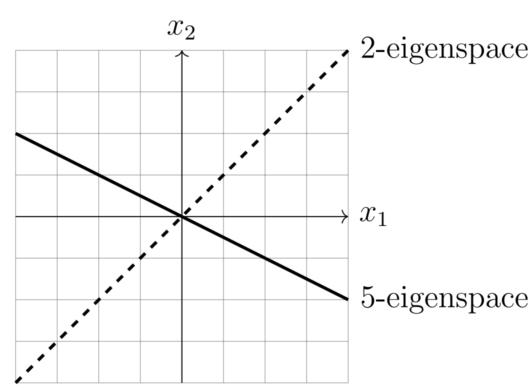
$$W = \begin{pmatrix} 4 & 1 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 4 & 2 \\ -2 & -1 & -1 & 1 \end{pmatrix}$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

(a) (5 points) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & 4 \end{pmatrix}$. Find A^{-1}. Write your answer in the space below.

$$A^{-1} = \begin{pmatrix} \quad & \quad & \quad \\ & & \\ & & \end{pmatrix}$$

(b) (5 points) Let A be the 2×2 matrix whose 5-eigenspace is the **solid** line below and whose 2-eigenspace is the **dashed** line below. Find $A\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.



This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.