Math 1553 Sample Midterm 3, Fall 2024 SOLUTIONS

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

This is a practice exam. It was compiled by modifying one or two past midterms. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

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1. For each statement, answer TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

In each statement, A is a matrix whose entries are **real numbers**.
Solutions are on the next page.

(a) Suppose A is a 3×3 matrix and there is some b in $\mathbf{R}^3$ so that the equation $Ax = b$ has exactly one solution. Then A must be invertible.

● True

○ False

(b) If A is an $n \times n$ matrix and $\det(A) = 0$, then $\lambda = 0$ must be an eigenvalue of A.

● True

○ False

(c) There is a 3×3 real matrix A whose eigenvalues are -1, 3, and $2 + i$.

○ True

● False

(d) Suppose A is a 4×4 matrix and its eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = 3, \quad \lambda_3 = 5, \quad \lambda_4 = 7.$$

Then A must be diagonalizable.

● True

○ False

(e) If A is a 5×5 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda(\lambda + 2)(\lambda - 4)^3,$$

then the null space of A must be a line.

● True

○ False

Solution:

(a) True: If $Ax = b$ has exactly one solution for some b, then $Ax = 0$ must have exactly one solution, thus A is invertible by the Invertible Matrix Theorem (or by a direct argument using pivots).

(b) True: If $\det(A) = 0$ then A is not invertible, which means $Ax = 0x$ has infinitely many solutions and therefore 0 is an eigenvalue of A. Alternatively, if $\det(A) = 0$ then A is not invertible so $A - 0I$ is not invertible, thus $\lambda = 0$ is an eigenvalue of A. This problem was inspired by a true/false question in the 5.1 Webwork #7.

(c) False. This was basically taken from #4 of the 5.5 Webwork. If $2 + i$ is an eigenvalue, then $2 - i$ would also be an eigenvalue, whereby A would have 4 different eigenvalues (-1, 3, $2 + i$, and $2 - i$) which is impossible for a 3×3 matrix.

(d) True. This is a quintessential diagonalization question. We know that eigenvectors for different eigenvalues are linearly independent. Since A has 4 different eigenvalues, this means we get 4 linearly independent eigenvectors in $\mathbf{R}^4$, therefore A is diagonalizable by the Diagonalization Theorem.

(e) True. The eigenvalue $\lambda = 0$ has algebraic multiplicity 1 in the characteristic polynomial. Since for any eigenvalue we know

$$\text{(alg. mult.)} \geq \text{(geo. mult.)} \geq 1,$$

this gives us $1 \geq \text{geo. mult.} \geq 1$ for $\lambda = 0$, thus $\lambda = 0$ has geometric multiplicity 1. In other words, the null space (i.e. the 0-eigenspace) is a line.

2. **Solutions are on the next page.**

(a) (2 points) Let $A = \begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$. Find A^{-1}. Clearly fill in the correct bubble below.

● $\frac{1}{10}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$ ○ $\frac{1}{10}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$ ○ $-\frac{1}{2}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}$

○ $\frac{1}{10}\begin{pmatrix} 4 & -3 \\ 2 & 1 \end{pmatrix}$ ○ $\frac{1}{10}\begin{pmatrix} 4 & 3 \\ -2 & 1 \end{pmatrix}$ ○ $-\frac{1}{2}\begin{pmatrix} 1 & -3 \\ 2 & 4 \end{pmatrix}$

(b) (3 points) Suppose A is an invertible matrix whose inverse is given by

$$A^{-1} = \begin{pmatrix} -1 & 2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}.$$

(i) Suppose b is a vector in $\mathbf{R}^3$. How many solutions will the equation $Ax = b$ have? Clearly fill in the correct bubble below.

○ None ● Exactly one ○ Infinitely many ○ Not enough info

(ii) Which **one** of the vectors below is a solution to $Ax = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$?

○ $\begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ ● $\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 0 \\ -1/3 \end{pmatrix}$

(c) (2 pts) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}$.

Clearly fill in the correct bubble below.

○ 1 ○ -1 ● 2 ○ -2

○ 4 ○ -4 ○ 8 ○ -8

(d) (3 points) Suppose A and B are 2×2 matrices satisfying

$$\det(A) = 6, \qquad \det(B) = -3.$$

Which of the following statements must be true? Clearly fill in the bubble for all that apply.

● AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

○ $\det(3B^{-1}) = -1$.

○ $A - 6I$ is not invertible.

Solution:

(a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ satisfies $ad - bc \neq 0$, then A is invertible and $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Here,

$$A^{-1} = \frac{1}{4(1)-(2)(-3)}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix} = \frac{1}{10}\begin{pmatrix} 1 & 3 \\ -2 & 4 \end{pmatrix}.$$

(b) A is invertible so $Ax = b$ is guaranteed to have exactly one solution. To solve $Ax = \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix}$ we have

$$x = A^{-1}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 & 2 & -2 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} = \begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}.$$

(c) To get from $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$ to $\begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}$, we first swap rows 1 and 3 to get

$$\begin{pmatrix} g & h & i \\ d & e & f \\ a & b & c \end{pmatrix}. \quad \text{The new determinant is } 1(-1) = -1.$$

Next, we multiply the new first row by -2 to get

$$\begin{pmatrix} -2g & -2h & -2i \\ d & e & f \\ a & b & c \end{pmatrix}. \quad \text{The new determinant is } (-1)(-2) = 2.$$

Finally, we do a row-replacement that doesn't change the determinant to get

$$\begin{pmatrix} 4a-2g & 4b-2h & 4c-2i \\ d & e & f \\ a & b & c \end{pmatrix}. \quad \text{Determinant is still 2.}$$

(d) (i) is true: $\det(A) \neq 0$ and $\det(B) \neq 0$, so both A and B are invertible and we know the classic formula $(AB)^{-1} = B^{-1}A^{-1}$.
(ii) is false because $\det(3B^{-1}) = 3^2 \det(B^{-1}) = 3^2(-1/3) = -3$.
(iii) is false in general, for example $A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ satisfies $\det(A) = 6$, however $A - 6I = \begin{pmatrix} -3 & 0 \\ 0 & -4 \end{pmatrix}$ which is certainly invertible.

3. **Solutions are on the next page.**

(a) Suppose A is an $n \times n$ matrix. Which **one** of the following statements is **not** correct?

○ An eigenvalue of A is a scalar λ such that $A - \lambda I$ is not invertible.

● An eigenvalue of A is a scalar λ such that $(A - \lambda I)v = 0$ has a solution.

○ An eigenvalue of A is a scalar λ such that $Av = \lambda v$ for a nonzero vector v.

○ An eigenvalue of A is a scalar λ such that $\det(A - \lambda I) = 0$.

(b) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = 7x$, and let A be the standard matrix for T, so $T\begin{pmatrix} x \\ y \end{pmatrix} = A\begin{pmatrix} x \\ y \end{pmatrix}$. In the blank below, write one eigenvector v in the (-1)-eigenspace of A.

$$\boxed{v = \begin{pmatrix} -7 \\ 1 \end{pmatrix}} \quad \text{or} \quad \boxed{v = \begin{pmatrix} 1 \\ -1/7 \end{pmatrix}}, \quad \text{etc.}$$

(c) (2 points) Let $A = \begin{pmatrix} -1 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Which **one** of the following describes the 1-eigenspace of A?

○ $\left\{\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} -1 \\ -4 \\ -6 \end{pmatrix}\right\}$

● $\text{Span}\left\{\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}\right\}$ ○ $\text{Span}\left\{\begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}\right\}$ ○ All of $\mathbf{R}^3$

(d) (4 points) Let $A = \begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ -2 & 3 \end{pmatrix}^{-1}$. Which of the following are true? Clearly mark all that apply.

● The eigenvalues of A are $1/2$ and 1.

○ For each vector x in $\mathbf{R}^2$, it is the case that $A^n x$ approaches $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ as n becomes large.

● $\text{Nul}(A - I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 3 \end{pmatrix}\right\}$.

● $A^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^{10}\begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

Solution:

(a) This was copied and pasted from #2 in the 5.1-5.2 Supplement. The answer is (ii), because an eigenvalue is a scalar λ so that $(A - \lambda I)v = 0$ has a **non-trivial** solution.

If λ is not an eigenvalue of A, then it still satisfies $(A - \lambda I)v = 0$ for $v = 0$.

(b) Reflection across the line $y = 7x$ has eigenvalues 1 and -1.

The 1-eigenspace is the line $y = 7x$ itself. The (-1)-eigenspace is the line through $(0,0)$ perpendicular to $y = 7x$, which is the line $y = -(1/7)x$. Therefore, the (-1)-eigenspace is the span of $\begin{pmatrix} 1 \\ -1/7 \end{pmatrix}$ or equivalently the span of $\begin{pmatrix} -7 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 7 \\ -1 \end{pmatrix}$ etc.

(c) $\left(A - I \mid 0 \right) = \left(\begin{array}{ccc|c} -2 & -4 & -6 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, so $x_1 + 2x_2 + 3x_3 = 0$ where x_2 and x_3 are free.

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_2 - 3x_3 \\ x_2 \\ x_3 \end{pmatrix} = x_2 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}.$$

(d) A has been diagonalized for us, so its eigenvalues are $\lambda_1 = 1/2$ and $\lambda_2 = 1$, with corresponding eigenvectors $v_1 = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$, respectively. We use these facts below.

(i) True, the eigenvalues are 1/2 and 1.

(ii) False: in fact, this is not even true for $x = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$.

$$A^n \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^n \begin{pmatrix} 4 \\ -2 \end{pmatrix} \text{ which approaches } \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{ as } n \text{ gets very large.}$$

(iii) True: $\text{Nul}(A - I)$ is by definition the 1-eigenspace of A, which we know is the span of $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$.

(iv) True: since $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ is a $(1/2)$-eigenvector, we have

$$A \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad A^2 \begin{pmatrix} 4 \\ -2 \end{pmatrix} = A \left(\frac{1}{2} \begin{pmatrix} 4 \\ -2 \end{pmatrix} \right) = \frac{1}{2} \cdot A \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^2 \begin{pmatrix} 4 \\ -2 \end{pmatrix}, \quad \text{etc.,}$$

$$A^{10} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \left(\frac{1}{2}\right)^{10} \begin{pmatrix} 4 \\ -2 \end{pmatrix}.$$

4. **Solutions are on the next page.**

(a) (2 points) Find all values of c so that $\lambda = 2$ is an eigenvalue of the matrix $A = \begin{pmatrix} 4 & -3 \\ 4 & c \end{pmatrix}$. Clearly fill in the correct bubble below.

○ $c = -3$ only   ● $c = -4$ only   ○ $c = 4$ only   ○ $c = -6$ only

○ All c except -4 ○ All c except -6 ○ All c except 6

(b) (4 points) Let $B = \begin{pmatrix} 1 & 4 \\ 2 & 5 \end{pmatrix}$ and define a matrix transformation by $T(x) = Bx$. Find the area of $T(S)$, where S is the triangle with vertices $(-1, 1)$, $(2, 3)$, and $(5, 2)$.

The area is $\frac{27}{2}$

(c) (4 points) Suppose A is a 3×3 matrix. Which of the following statements are true? Clearly circle all that apply.

○ If B is a 3×3 matrix that has the same reduced row echelon form as A, then the eigenvalues of B are the same as the eigenvalues of A.

● If $\lambda = 3$ is an eigenvalue of A, then the equation $Ax = 3x$ must have infinitely many solutions.

● If the equation $(A - 2I)x = 0$ has only the trivial solution, then 2 is not an eigenvalue of A.

● It is impossible for A to have 4 different eigenvalues.

Solution:

(a) $\left(A - 2I \mid 0\right) = \left(\begin{array}{cc|c} 2 & -3 & 0 \\ 4 & c-2 & 0 \end{array}\right)$ which is non-invertible precisely when its determinant is 0, so

$$2(c-2) + 12 = 0, \quad 2c = -8, \quad c = -4.$$

(b) Using base point $(-1, 1)$, the area of the triangle is

$$\frac{1}{2}\left|\det\begin{pmatrix} 3 & 6 \\ 2 & 1 \end{pmatrix}\right| = \frac{1}{2}(9) = \frac{9}{2}.$$

$$\text{Area}(T(S)) = |\det(B)|\text{Area}(S) = |5 - 8|\frac{9}{2} = \frac{27}{2}.$$

(c) (i) is false: for example $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$ have the same RREF but different eigenvalues.
(ii) is true directly from the definition of eigenvalue.
(iii) is true, because then $Ax = 2x$ has only the trivial solution.
(iv) is true: a 3×3 matrix has a degree 3 characteristic polynomial which has at most 3 roots, therefore A has at most 3 different eigenvalues.

5. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Consider the matrix $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & -1 & 3 \end{pmatrix}$.

(a) Find the characteristic polynomial of A and the eigenvalues of A.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(c) Is A diagonalizable? If yes, find an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$ and write them in the space provided below. If no, justify why A is not diagonalizable.

Solution:

(a) Cofactor expansion along the first column gives

$$\det(A - \lambda I) = \det \begin{pmatrix} 2-\lambda & 1 & -1 \\ 0 & -\lambda & 2 \\ 0 & -1 & 3-\lambda \end{pmatrix} = (2-\lambda)\Big[(-\lambda)(3-\lambda)+2\Big]$$

$$= (2-\lambda)\Big[\lambda^2 - 3\lambda + 2\Big] = -(\lambda-2)\Big[(\lambda-2)(\lambda-1)\Big] = -(\lambda-2)^2(\lambda-1).$$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$.

(b)

$$\left(A - I \mid 0 \right) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \xrightarrow[R_1=R_1+R_2,\text{ then } R_2=-R_2]{R_3=R_3-R_2} \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So $x_1 = -x_3$, $x_2 = 2x_3$, and x_3 is free. A basis for the 1-eigenspace is $\left\{ \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} \right\}$.

$$\left(A - 2I \mid 0 \right) = \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -2 & 2 & 0 \\ 0 & -1 & 1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So x_1 and x_3 are free and $x_2 = x_3$. A basis for the 2-eigenspace is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}. \right\}$

(c) We've found 3 linearly independent eigenvectors, so A is diagonalizable: $A = CDC^{-1}$ where

$$C = \begin{pmatrix} -1 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

6. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

(a) Let $A = \begin{pmatrix} 5 & -5 \\ 4 & -3 \end{pmatrix}$.

(i) (4 points) Find the complex eigenvalues of A. Fully simplify your answer.

Solution:

$$\begin{aligned} 0 &= \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - (5-3)\lambda + (-15+20) \\ &= \lambda^2 - 2\lambda + 5, \end{aligned}$$

so

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(5)(1)}}{2} = \frac{2 \pm \sqrt{-16}}{2} = \frac{2 \pm 4i}{2} = \boxed{1 \pm 2i}.$$

(ii) (3 points) For the eigenvalue with *positive* imaginary part, find one corresponding eigenvector v. Enter your answer in the space below.

Solution: $\left(A - (1+2i)I \mid 0\right)$ is

$$\left(\begin{array}{cc|c} 5-(1+2i) & -5 & 0 \\ 4 & -3-(1+2i) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 4-2i & -5 & 0 \\ 4 & -4-2i & 0 \end{array}\right) = \left(\begin{array}{cc|c} a & b & 0 \\ (*) & (*) & 0 \end{array}\right).$$

One eigenvector is $v = \begin{pmatrix} -b \\ a \end{pmatrix} = \boxed{\begin{pmatrix} 5 \\ 4-2i \end{pmatrix}}$ by the 2×2 eigenvector trick. Other answers possible, like $v = \begin{pmatrix} -5 \\ -4+2i \end{pmatrix}$ or $v = \begin{pmatrix} 4+2i \\ 4 \end{pmatrix}$ or even $v = \begin{pmatrix} \frac{2+i}{2} \\ 1 \end{pmatrix}$, etc.

(b) (3 points) Given that

$$\det\begin{pmatrix} 0 & -1 & 3 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 28, \quad \det\begin{pmatrix} 4 & 2 & -1 \\ 0 & 4 & 2 \\ -2 & -1 & 1 \end{pmatrix} = 8, \quad \text{and } \det\begin{pmatrix} 4 & 2 & -1 \\ 0 & -1 & 3 \\ 0 & 4 & 2 \end{pmatrix} = -56,$$

compute the determinant of the 4×4 matrix W below.

$$W = \begin{pmatrix} 4 & 1 & 2 & -1 \\ 0 & 2 & -1 & 3 \\ 0 & 0 & 4 & 2 \\ -2 & -1 & -1 & 1 \end{pmatrix}$$

Solution: By cofactor expansion along the second column,

$$\begin{aligned} \det(W) &= 1C_{12} + 2C_{22} + 0C_{32} - 1C_{42} \\ &= 1(-1)^3(28) + 2(-1)^4(8) + 0 - 1(-1)^6(-56) \\ &= -28 + 16 + 56 = \boxed{44}. \end{aligned}$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

(a) (5 points) Let $A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 1 & 0 & 4 \end{pmatrix}$. Find A^{-1}. Write your answer in the space below.

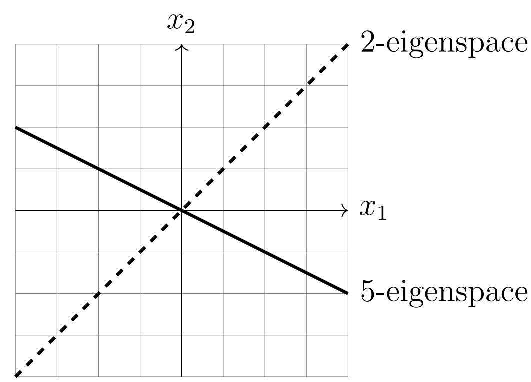
$A^{-1} = \begin{pmatrix} 4 & 0 & -3 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{pmatrix}$

Solution:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 1 & 0 & 4 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_3 = R_3 - R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = -R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & & \\ 0 & 1 & -2 & 0 & & \\ 0 & 0 & 1 & -1 & & \end{array}\right)$$

$$\xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 + 2R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & 0 & -3 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right)$$

(b) (5 points) Let A be the 2×2 matrix whose 5-eigenspace is the **solid** line below and whose 2-eigenspace is the **dashed** line below. Find $A\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.



Solution: The 2-eigenspace is spanned by $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ and the 5-eigenspace is spanned by $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$. Also, note $\begin{pmatrix} 4 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}$, therefore

$$A\begin{pmatrix} 4 \\ 1 \end{pmatrix} = A\left(\begin{pmatrix} 2 \\ 2 \end{pmatrix} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\right) = A\begin{pmatrix} 2 \\ 2 \end{pmatrix} + A\begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

$$= 2\begin{pmatrix} 2 \\ 2 \end{pmatrix} + 5\begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} + \begin{pmatrix} 10 \\ -5 \end{pmatrix} = \begin{pmatrix} 14 \\ -1 \end{pmatrix}.$$

Alternatively, we could have computed

$$
\begin{aligned}
A\begin{pmatrix}4\\1\end{pmatrix} &= \begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}^{-1}\begin{pmatrix}4\\1\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}-1 & -2\\-2 & 2\end{pmatrix}\begin{pmatrix}4\\1\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}2 & 0\\0 & 5\end{pmatrix}\begin{pmatrix}-6\\-6\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}2 & 2\\2 & -1\end{pmatrix}\begin{pmatrix}-12\\-30\end{pmatrix}\\
&= -\frac{1}{6}\begin{pmatrix}-84\\6\end{pmatrix} = \begin{pmatrix}14\\-1\end{pmatrix}.
\end{aligned}
$$

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.