Math 1553 Sample Midterm 2, Fall 2024

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15)   Wessels(B, 8:25-9:15)   Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20)   Kim (G, 12:30-1:20)   Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50)   He (L, 3:30-4:20)   Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50)   Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles to fill in, you must fill them in the correct bubbles clearly and completely or you will not receive credit.

This is a practice exam. It was compiled by modifying one or two past midterms. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) Suppose v_1, v_2, and b are vectors in $\mathbf{R}^n$ and the equation

   $$x_1 v_1 + x_2 v_2 = b$$

   has exactly one solution. Then $\{v_1, v_2, b\}$ must be linearly independent.

   ○ True

   ○ False

   (b) Let V be the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ of the form

   $$x - 3y + z = 2.$$

   Then V is a subspace of $\mathbf{R}^3$.

   ○ True

   ○ False

   (c) If a vector x is in the null space of a matrix A, then $4x$ is also in the null space of A.

   ○ True

   ○ False

   (d) If A is a 7×4 matrix, then the matrix transformation $T(x) = Ax$ cannot be onto.

   ○ True

   ○ False

   (e) If A is a 4×3 matrix and the equation $Ax = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ has exactly one solution, then the transformation $T(x) = Ax$ must be one-to-one.

   ○ True

   ○ False

2. Multiple choice and short answer. You do not need to show work or justify your answers. Parts (a), (b), and (c) are unrelated.

(a) (3 points) Suppose v_1, v_2, v_3, and v_4 are vectors in $\mathbf{R}^4$. Which of the following must be true? Clearly circle all that apply.

◯ If $\text{Span}\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4$, then the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

◯ Suppose that the matrix A with columns v_1 through v_4 has one pivot, and let T be the matrix transformation $T(x) = Ax$. Then the range of T is a line.

◯ If $v_1 - v_2 - v_3 + v_4 = 0$, then $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

(b) (4 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \ : \ y \leq 3x \right\}$.

Answer each of the following questions.

(i) Does V contain the zero vector? ◯ YES ◯ NO

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? ◯ YES ◯ NO

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? ◯ YES ◯ NO

(iv) Is V a subspace of $\mathbf{R}^2$? ◯ YES ◯ NO

(c) (3 pts) Let A be a 15×7 matrix. If $\text{rank}(A) \leq 3$, which of the following are possible values for $\text{nullity}(A)$? Clearly select all that apply.

◯ 1

◯ 3

◯ 4

◯ 6

◯ 7

◯ 9

3. Short answer. Parts (a) through (d) are unrelated. There is no work necessary on this problem.

(a) This part is worth 4 points total.

(i) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation for **counterclockwise** rotation by 13°. Find the standard matrix A for the transformation T.

○ $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ \sin(13°) & \cos(13°) \end{pmatrix}$ ○ $A = \begin{pmatrix} \cos(13°) & \sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$

○ $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$ ○ $A = \begin{pmatrix} -\cos(13°) & -\sin(13°) \\ \sin(13°) & -\cos(13°) \end{pmatrix}$

(ii) Write a specific matrix B satisfying $B\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-10\\2\end{pmatrix}$. Is this the only matrix satisfying $B\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-10\\2\end{pmatrix}$?

(b) (3 points) Let A be an $m \times n$ matrix, and let T be the associated linear transformation $T(x) = Ax$. Which of the following statements are true? Clearly circle all that apply.

○ If the columns of A are linearly independent, then $\dim(\text{range}(T)) = n$.

○ If the columns of A are linearly dependent, then T is onto.

○ If T is onto, then $\text{Col}(A) = \mathbf{R}^m$.

(c) (3 points) Determine which of the following statements are true.

○ If A is an $m \times n$ matrix, then there is a matrix C so that $A + C = 0$.

○ If A is a 3×7 matrix and B is a 4×3 matrix, then the matrix transformation T given by $T(x) = BAx$ has domain $\mathbf{R}^4$ and codomain $\mathbf{R}^7$.

○ If A is an $n \times n$ matrix and $A^2 = 0$, then $(I - A)(I + A) = I$.

(d) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first rotates vectors by 90 degrees clockwise, then reflects across the line $y = -x$. Find $T\begin{pmatrix}0\\2\end{pmatrix}$. Clearly circle your answer below.

○ $\begin{pmatrix}0\\2\end{pmatrix}$ ○ $\begin{pmatrix}0\\-2\end{pmatrix}$ ○ $\begin{pmatrix}2\\0\end{pmatrix}$

○ $\begin{pmatrix}-2\\0\end{pmatrix}$ ○ $\begin{pmatrix}-1\\1\end{pmatrix}$ ○ $\begin{pmatrix}1\\-1\end{pmatrix}$

4. You do not need to show your work on this problem. Parts (a), (b), (c), and (d) are unrelated.

(a) (3 points) Suppose that A is a matrix that represents a linear transformation T from $\mathbf{R}^7$ to $\mathbf{R}^9$. In other words, T is the transformation given by the formula $T(x) = Ax$.

(i) How many rows does the matrix A have? Enter your answer here: ____________.

(ii) Suppose the reduced row echelon form of the matrix A contains 3 pivots. Apply the Rank Theorem to A to fill in the following blanks with numbers.

$\dim(\text{Col } A) =$ ________ $\qquad$ $\dim(\text{Nul } A) =$ ________.

(b) (2 points) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation. Which **one** of the following is a definition that T is one-to-one?

○ For each x in $\mathbf{R}^n$, there is a vector y in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each x in $\mathbf{R}^n$, there is at most one vector y in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each y in $\mathbf{R}^m$, there is at most one vector x in $\mathbf{R}^n$ so that $T(x) = y$.

(c) (3 points) Let V be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form

$$\begin{pmatrix} -4x_4 \\ x_2 \\ x_2 + 6x_4 \\ x_4 \end{pmatrix}.$$

Write a basis for V.

(d) (2 points) Which of the following linear transformations are one-to-one? Clearly circle all that apply.

○ $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates vectors counterclockwise by 15°.

○ $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} x.$

5. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. For this problem, consider the matrix A and its reduced row echelon form given below.

$$A = \begin{pmatrix} 1 & 7 & 0 & -4 \\ -1 & -7 & 1 & 7 \\ 2 & 14 & 1 & -5 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 7 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (2 points) Write a basis for Col A. Briefly justify your answer.

(b) (4 points) Find a basis for Nul A.

(c) (2 points) Write one vector x that is not the zero vector and is in the null space of A. Briefly justify your answer.

(d) (2 points) Let T be the matrix transformation $T(x) = Ax$. Circle the correct answers below. You do not need to show your work on this part.

(i) The range of T is:

a point   a line   a plane   all of $\mathbf{R}^3$ all of $\mathbf{R}^4$

(ii) The range of T is a subspace of:

$\mathbf{R}$ $\mathbf{R^2}$ $\mathbf{R}^3$ $\mathbf{R}^4$

6. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation by $90°$ counterclockwise.
Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = x$.
Let $V : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation

$$V(x_1, x_2, x_3) = (x_1 - 5x_2,\ 3x_1 - 4x_3).$$

(a) (2 points) Write the standard matrix A for T.
(do *not* leave your answer in terms of sine and cosine; simplify it completely)

(b) (2 points) Write the standard matrix B for U.

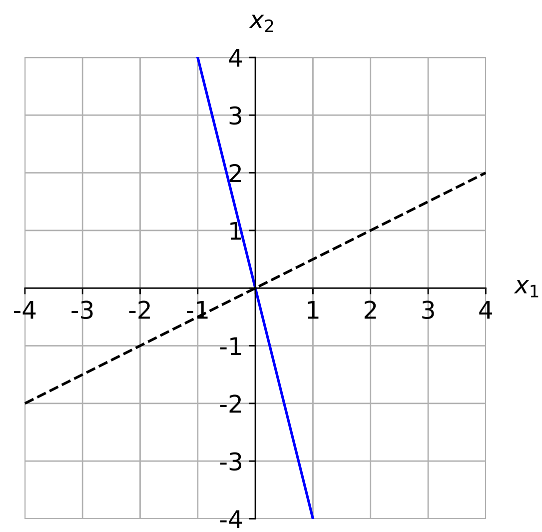
(c) (3 points) Find the standard matrix C for V.

(d) (3 points) Find the standard matrix D for the transformation $W : \mathbf{R}^2 \to \mathbf{R}^2$ that first reflects vectors in $\mathbf{R}^2$ across the line $y = x$, then rotates vectors by $90°$ counterclockwise.

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a), (b), and (c) are unrelated.

(a) (4 points) Let $A = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix}$.
Find all values of a and b so that $A^2 = A$.

(b) (4 points) Write a single matrix A with the property that $\mathrm{Col}(A)$ is the solid line graphed below and $\mathrm{Nul}(A)$ is the dotted line graphed below.



(c) (2 points) Give one specific example of a subspace V of $\mathbf{R}^3$ that contains the vector $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Briefly justify your answer.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.