# Math 1553 Sample Midterm 2, Fall 2024 SOLUTIONS

| **Name** | | **GT ID** | |
|---|---|---|---|

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page.
- The maximum score on this exam is 70 points, and you have 75 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- As always, RREF means "reduced row echelon form."
- The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles to fill in, you must fill them in the correct bubbles clearly and completely or you will not receive credit.

This is a practice exam. It was compiled by modifying one or two past midterms. It is meant to be similar in format, length, and difficulty to the real exam. It is **not** meant as a comprehensive list of study problems. We recommend completing the practice exam in 75 minutes, without notes or distractions.

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1. TRUE or FALSE. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

   (a) Suppose $v_1, v_2$, and $b$ are vectors in $\mathbf{R}^n$ and the equation

   $$x_1 v_1 + x_2 v_2 = b$$

   has exactly one solution. Then $\{v_1, v_2, b\}$ must be linearly independent.

   ○ True

   ● False

   (b) Let $V$ be the set of all vectors $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in $\mathbf{R}^3$ of the form

   $$x - 3y + z = 2.$$

   Then $V$ is a subspace of $\mathbf{R}^3$.

   ○ True

   ● False

   (c) If a vector $x$ is in the null space of a matrix $A$, then $4x$ is also in the null space of $A$.

   ● True

   ○ False

   (d) If $A$ is a $7 \times 4$ matrix, then the matrix transformation $T(x) = Ax$ cannot be onto.

   ● True

   ○ False

   (e) If $A$ is a $4 \times 3$ matrix and the equation $Ax = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$ has exactly one solution, then the transformation $T(x) = Ax$ must be one-to-one.

   ● True

   ○ False

**Solution:**

(a) False. The fact that $x_1v_1+x_2v_2 = b$ is consistent means that $b$ is in Span$\{v_1, v_2\}$, so the set $\{v_1, v_2, b\}$ is automatically linearly dependent by the Increasing Span Criterion.

(b) False. If we check the conditions for being a subspace, we find that $V$ immediately fails the first property: $V$ does not contain the zero vector since $0 - 3(0) + 0 \neq 2$. In reality, $V$ is a plane in $\mathbf{R}^3$, but it does not have any of the three properties of a subspace.

(c) True: if $A$ is any $m \times n$ matrix then Nul $A$ is a subspace of $\mathbf{R}^n$ and therefore Nul $A$ is closed under scalar multiplication, so if $x$ is in Nul $A$ then so is every scalar multiple of $x$. Alternatively, we could have just used computations rather than the theory of subspaces. If $x$ is in the null space of $A$, then $Ax = 0$, so $A(4x) = 4Ax = 4(0) = 0$.

(d) True: the $7 \times 4$ matrix $A$ has 7 rows but a maximum of 4 pivots. Therefore, $A$ cannot have a pivot in every row, so $T$ cannot be onto.

(e) True. If $Ax = b$ has a unique solution for some $b$, then $Ax = 0$ has exactly one solution, which means that $A$ has a pivot in every column, so $T$ is one-to-one.

2. **Full solutions are on the next page.**

(a) (3 points) Suppose $v_1, v_2, v_3$, and $v_4$ are vectors in $\mathbf{R}^4$. Which of the following must be true? Clearly circle all that apply.

● If $\text{Span}\{v_1, v_2, v_3, v_4\} = \mathbf{R}^4$, then the set $\{v_1, v_2, v_3, v_4\}$ is linearly independent.

● Suppose that the matrix $A$ with columns $v_1$ through $v_4$ has one pivot, and let $T$ be the matrix transformation $T(x) = Ax$. Then the range of $T$ is a line.

● If $v_1 - v_2 - v_3 + v_4 = 0$, then $\{v_1, v_2, v_3, v_4\}$ is linearly dependent.

(b) (4 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \ : \ y \leq 3x \right\}$.

Answer each of the following questions.

(i) Does $V$ contain the zero vector? ● YES ○ NO

(ii) Is $V$ closed under addition? In other words, if $u$ and $v$ are in $V$, must it be true that $u + v$ is in $V$? ● YES ○ NO

(iii) Is $V$ closed under scalar multiplication? In other words, if $c$ is a real number and $u$ is in $V$, must it be true that $cu$ is in $V$? ○ YES ● NO

(iv) Is $V$ a subspace of $\mathbf{R}^2$? ○ YES ● NO

(c) (3 pts) Let $A$ be a $15 \times 7$ matrix. If $\text{rank}(A) \leq 3$, which of the following are possible values for $\text{nullity}(A)$? Clearly select all that apply.

○ 1

○ 3

● 4

● 6

● 7

○ 9

**Solution: Problem 2**

(a) (i) True by the Basis Theorem: if the span of the 4 vectors is $\mathbf{R}^4$, then the vectors are automatically linearly independent. Alternatively, we could reason without using the Basis Theorem. if the span of those 4 vectors is $\mathbf{R}^4$, then the matrix with columns $v_1$ through $v_4$ will have 4 pivots, thus $A$ has a pivot in every column and so the vectors are linearly independent.

(ii) True: the range of $T$ is the column space of $A$, which is a line since $A$ has one pivot.

(iii) True directly by the definition of linearly independence. In fact, the equation $v_1 - v_2 - v_3 + v_4 = 0$ is a linear dependence relation for the vectors.

(b) We can draw $V$ and see right away that it is not a subspace of $\mathbf{R}^2$, since it is neither the zero vector nor a line through the origin nor the entirety of $\mathbf{R}^2$. The set $V$ consists of all points on, and below, the line $y = 3x$.
(i) Yes: $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ satisfies $y \leq 3x$ since $0 \leq 0$.

(ii) Yes: we can see geometrically that if we add any vectors in the region $V$, the sum will stay in $V$. We could have done it algebraically instead, those this is not as easy. If $\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$ and $\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$ are in $V$ (so $y_1 \leq 3x_1$ and $y_2 \leq 3x_2$) then $\begin{pmatrix} x_1 + x_2 \\ y_1 + y_2 \end{pmatrix}$ is in $V$ since

$$y_1 + y_2 \leq 3x_1 + 3x_2 = 3(x_1 + x_2).$$

(iii) No. We can see geometrically that if $u$ is a nonzero vector in $V$, then $-u$ is often not in $V$. Alternatively, we can see this geometrically. For example, take $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. We see $u$ is in $V$ since $0 \leq 3(1)$. However, $-u = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is not in $V$ since $0 > -3$.

(iv) No: column spaces are always subspaces, but $V$ is not a subspace, so $V$ cannot be the column space of any matrix. Alternatively, without even doing (i)-(iii) we could do this using spans: if $V$ were the span of the columns of some matrix, then since $V$ contains $(1, 0)$ and $(0, -1)$ (since they satisfy $y \leq 3x$) it would need contain their span, which is all of $\mathbf{R}^2$. But this is nonsense, since $V$ is clearly not $\mathbf{R}^2$.

(c) This is a classic Rank Theorem problem. By the Rank Theorem we know

$$\text{rank}(A) + \text{nullity}(A) = 7, \quad \text{so} \quad \text{nullity}(A) = 7 - \text{rank}(A).$$

From the fact that $0 \leq \text{rank}(A) \leq 3$, we know that nullity$(A)$ must be at least 4 but cannot be greater than 7.

3. **Full solutions are on the next page.**

(a) This part is worth 4 points total.

(i) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation for **counterclockwise** rotation by 13°. Find the standard matrix $A$ for the transformation $T$.

● $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ \sin(13°) & \cos(13°) \end{pmatrix}$  ○ $A = \begin{pmatrix} \cos(13°) & \sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$

○ $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ -\sin(13°) & \cos(13°) \end{pmatrix}$  ○ $A = \begin{pmatrix} -\cos(13°) & -\sin(13°) \\ \sin(13°) & -\cos(13°) \end{pmatrix}$

(ii) Write a specific matrix $B$ satisfying $B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 2 \end{pmatrix}$. Is this the only matrix satisfying $B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 2 \end{pmatrix}$?

Many matrices $B$ have this property, for example $\boxed{B = \begin{pmatrix} -10 & 0 \\ 2 & 0 \end{pmatrix}}$.

(b) (3 points) Let $A$ be an $m \times n$ matrix, and let $T$ be the associated linear transformation $T(x) = Ax$. Which of the following statements are true? Clearly circle all that apply.

● If the columns of $A$ are linearly independent, then $\dim(\text{range}(T)) = n$.

○ If the columns of $A$ are linearly dependent, then $T$ is onto.

● If $T$ is onto, then $\text{Col}(A) = \mathbf{R}^m$.

(c) (3 points) Determine which of the following statements are true.

● If $A$ is an $m \times n$ matrix, then there is a matrix $C$ so that $A + C = 0$.

○ If $A$ is a $3 \times 7$ matrix and $B$ is a $4 \times 3$ matrix, then the matrix transformation $T$ given by $T(x) = BAx$ has domain $\mathbf{R}^4$ and codomain $\mathbf{R}^7$.

● If $A$ is an $n \times n$ matrix and $A^2 = 0$, then $(I - A)(I + A) = I$.

(d) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first rotates vectors by 90 degrees clockwise, then reflects across the line $y = -x$. Find $T\begin{pmatrix} 0 \\ 2 \end{pmatrix}$. Clearly circle your answer below.

○ $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$  ● $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$  ○ $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

○ $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$  ○ $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$  ○ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

**Solution: Problem 3**

(a) (i) Counterclockwise rotation by angle $\theta$ is given by $\begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, so the answer is (I): $A = \begin{pmatrix} \cos(13°) & -\sin(13°) \\ \sin(13°) & \cos(13°) \end{pmatrix}$

(ii) The condition $B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 2 \end{pmatrix}$ means that the transformation $U(x) = Bx$ must have domain $\mathbf{R}^2$ and codomain $\mathbf{R}^2$, so $B$ must be $2 \times 2$. If we call its columns $b_1$ and $b_2$, then the condition $B \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 2 \end{pmatrix}$ just means $1b_1 + 1b_2 = \begin{pmatrix} -10 \\ 2 \end{pmatrix}$, so we just need the sum of the columns of $B$ to be equal to $\begin{pmatrix} -10 \\ 2 \end{pmatrix}$. Infinitely many examples are possible. For example:

$$B = \begin{pmatrix} -10 & 0 \\ 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -10 \\ 0 & 2 \end{pmatrix},$$

$$B = \begin{pmatrix} -5 & -5 \\ 1 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} -10 & 0 \\ 0 & 2 \end{pmatrix}, \quad \text{etc.}$$

(b) (i) True: if the columns of $A$ are linearly independent, then all $n$ columns have pivots and are therefore pivot columns, so $\dim(\mathrm{Col}(A)) = n$.

(ii) Not necessarily true. For example, take $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

(iii) True: in fact this is nearly one of the definitions of "onto."

(c) (i) True, in fact $C = -A$.

(ii) Not true: $BA$ will be a $4 \times 7$ matrix, so $T$ will have domain $\mathbf{R}^7$ and codomain $\mathbf{R}^4$.

(iii) True:

$$(I - A)(I + A) = I^2 + A - A - A^2 = I - A^2 = I - 0 = I.$$

(d) We rotate $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ clockwise by 90 degrees to get $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, then we reflect $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ across the line $y = -x$ to get $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$. Alternatively, we could do matrix multiplication instead. Since we are doing the rotation first and then the reflection second, we must put the rotation matrix on the right and the reflection on the left (this is not a typo):

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

so our answer is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$.

4. You do not need to show your work on this problem. Parts (a), (b), (c), and (d) are unrelated. Solutions are on the next page.

(a) (3 points) Suppose that $A$ is a matrix that represents a linear transformation $T$ from $\mathbf{R}^7$ to $\mathbf{R}^9$. In other words, $T$ is the transformation given by the formula $T(x) = Ax$.

(i) How many rows does the matrix $A$ have? Enter your answer here: 9

(ii) Suppose the reduced row echelon form of the matrix $A$ contains 3 pivots. Apply the Rank Theorem to $A$ to fill in the following blanks with numbers.

$$\dim(\text{Col } A) = \underline{\ \ 3\ \ } \qquad \dim(\text{Nul } A) = \underline{\ \ 4\ \ }.$$

(b) (2 points) Suppose $T : \mathbf{R}^n \to \mathbf{R}^m$ is a transformation. Which **one** of the following is a definition that $T$ is one-to-one?

○ For each $x$ in $\mathbf{R}^n$, there is a vector $y$ in $\mathbf{R}^m$ so that $T(x) = y$.

○ For each $x$ in $\mathbf{R}^n$, there is at most one vector $y$ in $\mathbf{R}^m$ so that $T(x) = y$.

● For each $y$ in $\mathbf{R}^m$, there is at most one vector $x$ in $\mathbf{R}^n$ so that $T(x) = y$.

(c) (3 points) Let $V$ be the subspace of $\mathbf{R}^4$ consisting of all vectors of the form

$$\begin{pmatrix} -4x_4 \\ x_2 \\ x_2 + 6x_4 \\ x_4 \end{pmatrix}.$$

Write a basis for $V$.

Many answers possible, for example $\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 6 \\ 1 \end{pmatrix} \right\}$.

(d) (2 points) Which of the following linear transformations are one-to-one? Clearly circle all that apply.

● $T : \mathbf{R}^2 \to \mathbf{R}^2$ that rotates vectors counterclockwise by 15°.

○ $T : \mathbf{R}^3 \to \mathbf{R}^3$ given by $T(x) = \begin{pmatrix} 1 & 2 & 2 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix} x$.

**Solution:**

(a) (i) $T$ has domain $\mathbf{R}^7$ and codomain $\mathbf{R}^9$, so $A$ is $9 \times 7$, therefore $A$ has 9 rows.
(ii) $A$ has 3 pivots, so $\dim(\text{Col } A) = 3$. By the Rank Theorem,

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 7, \qquad \text{so} \qquad \dim(\text{Nul } A) = 4.$$

(b) Statement (iii) says that $T$ is one-to-one.
Statement (i) just says $T$ is a transformation. Statement (ii) is a slight modification of the definition of transformation, and in fact if some $x$ in $\mathbf{R}^n$ did not have a $y$ so that $T(x) = y$, then $T$ would fail to be a transformation in the first place because the domain of $T$ would no longer be all of $\mathbf{R}^n$.

(c) We see $V$ is the set of all vectors of the form $x_2 \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -4 \\ 0 \\ 6 \\ 1 \end{pmatrix}$, so one basis is

$$\left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 6 \\ 1 \end{pmatrix} \right\}.$$

Other answers are possible.

(d) (i) Yes, $T$ is one-to-one. In fact, it is both one-to-one and onto!
(ii) No, $T$ is not one-to-one. One step of row-reduction shows that $A$ only has 2 pivots, so $A$ has a column without a pivot.

5. For this problem, consider the matrix $A$ and its reduced row echelon form given below.

$$A = \begin{pmatrix} 1 & 7 & 0 & -4 \\ -1 & -7 & 1 & 7 \\ 2 & 14 & 1 & -5 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 7 & 0 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) (2 points) Write a basis for Col $A$. Briefly justify your answer.

**Solution**: The first and third columns of $A$ are pivot columns, so they form a basis for Col $A$. In reality, **any choice of two columns** of $A$ *except for the first two* will be linearly independent and thus a basis for Col $A$.

$$\text{One possible answer: } \left\{ \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}.$$

(b) (4 points) Find a basis for Nul $A$.

**Solution**: The RREF of $(A|0)$ gives homogeneous solution set $x_1 + 7x_2 - 4x_4 = 0$ and $x_3 = -3x_4$ where $x_2$ and $x_4$ are free. Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -7x_2 + 4x_4 \\ x_2 \\ -3x_4 \\ x_4 \end{pmatrix} = x_2 \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}. \text{ Basis for Nul A : } \left\{ \begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \right\}.$$

(c) (2 points) Write one vector $x$ that is not the zero vector and is in the null space of $A$. Briefly justify your answer.

Any nonzero linear combination of $\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix}$ is correct. For example,

$$\begin{pmatrix} -7 \\ 1 \\ 0 \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} 4 \\ 0 \\ -3 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -3 \\ 1 \\ -3 \\ 1 \end{pmatrix}, \quad \text{etc.}$$

(d) (2 points) Let $T$ be the matrix transformation $T(x) = Ax$. Circle the correct answers below. You do not need to show your work on this part.

(i) The range of $T$ is:

a point | a line | [a plane] | all of $\mathbf{R}^3$ | all of $\mathbf{R}^4$

(ii) The range of $T$ is a subspace of:

$\mathbf{R}$ | $\mathbf{R}^2$ | [$\mathbf{R}^3$] | $\mathbf{R}^4$

6. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation of rotation by $90°$ counterclockwise.
Let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the line $y = x$.
Let $V : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation

$$V(x_1, x_2, x_3) = (x_1 - 5x_2,\ 3x_1 - 4x_3).$$

(a) (2 points) Write the standard matrix $A$ for $T$.
(do *not* leave your answer in terms of sine and cosine; simplify it completely)
**Solution**: $A = \begin{pmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

(b) (2 points) Write the standard matrix $B$ for $U$.

**Solution**: $U \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix}$, so

$$B = \left( U\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad U\begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

(c) (3 points) Find the standard matrix $C$ for $V$.
**Solution**: $C = \left( V\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad V\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad V\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 1 & -5 & 0 \\ 3 & 0 & -4 \end{pmatrix}$

(d) (3 points) Find the standard matrix $D$ for the transformation $W : \mathbf{R}^2 \to \mathbf{R}^2$ that first reflects vectors in $\mathbf{R}^2$ across the line $y = x$, then rotates vectors by $90°$ counterclockwise.
**Solution**: Since we are doing $U$ first and then $T$, our composition is $T \circ U$, so the matrix is $D = AB$.

$$D = AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Alternatively, we could just follow the steps for $W\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $W\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

$$W\begin{pmatrix} 1 \\ 0 \end{pmatrix} : \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

$$W\begin{pmatrix} 0 \\ 1 \end{pmatrix} : \begin{pmatrix} 0 \\ 1 \end{pmatrix} \xrightarrow{\text{reflect}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \xrightarrow{\text{rotate}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit. Parts (a), (b), and (c) are unrelated.

(a) (4 points) Let $A = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix}$.

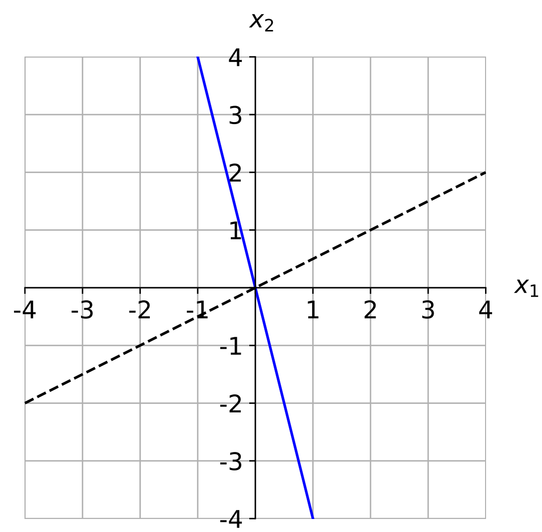
Find all values of $a$ and $b$ so that $A^2 = A$.

**Solution**: $\boxed{a = 12}$ and $\boxed{b = -3}$.

$$A^2 = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix} \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix} = \begin{pmatrix} 16-a & 4a+ab \\ -4-b & -a+b^2 \end{pmatrix}, \qquad A = \begin{pmatrix} 4 & a \\ -1 & b \end{pmatrix}.$$

Setting $A^2 = A$ gives $16 - a = 4$ in the "11" entry, so $a = 12$. Now, in the "21" entry we have $-4 - b = -1$, so $b = -3$. One can also check that the other two entries are satisfied: $4a + ab = a$ since $48 - 3(12) = 12$, and $-a + b^2 = b$ since $-12 + (-3)^2 = -3$.

(b) (4 points) Write a single matrix $A$ with the property that Col($A$) is the solid line graphed below and Nul($A$) is the dotted line graphed below.




**Solution**: We need $\text{Col}(A) = \text{Span}\left\{\begin{pmatrix} 1 \\ -4 \end{pmatrix}\right\}$ and $\text{Nul}(A) = \text{Span}\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$, so each column of $A$ must be a multiple of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and $A\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ must equal $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ (in other words, the homogeneous system $Ax = 0$ will have parametric form $x_1 = 2x_2$ where $x_2$ is free, thus $x_1 - 2x_2 = 0$).

A correct answer $A$ must have each column a multiple of $\begin{pmatrix} 1 \\ -4 \end{pmatrix}$ and the second column must be $-2$ times the first. For example,

$$A = \begin{pmatrix} 1 & -2 \\ -4 & 8 \end{pmatrix}, \quad A = \begin{pmatrix} -1 & 2 \\ 4 & -8 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & -4 \\ -8 & 16 \end{pmatrix}, \quad A = \begin{pmatrix} 1/2 & -1 \\ -2 & 4 \end{pmatrix}, \quad \text{etc.}$$

(c) (2 points) Give one specific example of a subspace $V$ of $\mathbf{R}^3$ that contains the vector $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Briefly justify your answer.

**Solution**: Note that the single vector $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$ by itself is NOT a subspace of $\mathbf{R}^3$ since it does not contain the zero vector, is not closed under addition, and is not closed under scalar multiplication!

Many answers are possible. For example, $V = \text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$ is a subspace that contains $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$. Another such subspace is the $xy$-plane of $\mathbf{R}^3$, or in other words

$$V = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Another answer is $V = \mathbf{R}^3$, since $\mathbf{R}^3$ is a subspace of itself and $\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ is certainly in $\mathbf{R}^3$.

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