Math 1553 Final Examination, SOLUTIONS, Fall 2024

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- Simplify all fractions as much as possible. As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until 8:50 PM on Tuesday, December 10.

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1. (2 pts each) **Solutions are on the next page.**

(a) If A is an $n \times n$ matrix and the equation $Ax = 0$ has only the trivial solution, then $\det(A) = 0$.

○ True ● False

(b) If $\{u, v, w\}$ is a linearly independent set of vectors in $\mathbf{R}^n$, then the set $\{u,\ u - v,\ u - v + 2w\}$ must also be linearly independent.

● True ○ False

(c) If W is a subspace of $\mathbf{R}^n$, then there is at least one matrix A so that $\text{Col}(A) = W$.

● True ○ False

(d) There is a 2×2 positive stochastic matrix that has $-\frac{i}{2}$ as an eigenvalue.

○ True ● False

(e) Let A be a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = \lambda^2(1 - \lambda).$$

If A is diagonalizable, then the RREF of A must have exactly one pivot.

● True ○ False

Solution: Problem 1

(a) False: $Ax = 0$ has only the trivial solution, so A is invertible, which means $\det(A) \neq 0$.

(b) True, by the Increasing Span Criterion: $u - v$ is not in the span of u, and $u - v + 2w$ is not in $\text{Span}\{u, u - v\}$, so the set $\{u, u - v, u - v + 2w\}$ is linearly independent.

(c) True: W is a subspace, so W is a span and therefore $W = \text{Span}\{v_1, \dots, v_k\}$ for some vectors $v_1, \dots, v_k$.
Therefore, $W = \text{Col}(A)$ for the matrix A whose columns are $v_1, \dots, v_k$.

(d) False. Every 2×2 positive stochastic matrix has $\lambda = 1$ as an eigenvalue. If $-i/2$ is an eigenvalue, then $i/2$ must also be an eigenvalue by the theory of complex eigenvalues. This would make A a 2×2 matrix with 3 different eigenvalues, which is impossible since an $n \times n$ matrix can have at most n different eigenvalues.

(e) True. We note the eigenvalues are $\lambda = 0$ and $\lambda = 1$. By the Diagonalization Theorem, A is diagonalizable if and only if it admits 3 linearly independent eigenvectors.

- The geometric multiplicity of $\lambda = 1$ is automatically 1 since its algebraic multiplicity is 1.
- The geometric multiplicity of $\lambda = 0$ must therefore be 2 for A to be diagonalizable. Since the 0-eigenspace is just the null space of A, this means the null space of A is a plane (the homogeneous equation $Ax = 0$ has two free variables), so when we row-reduce A we will find it has exactly one pivot.

2. Multiple choice. You do not need to show your work on this page, and there is no partial credit.

   (a) (2 points) Consider the following sets in $\mathbf{R}^2$. Clearly fill in the bubble for the **one** set V that satisfies **all** of the following conditions.

   (1) V contains the zero vector.

   (2) V is *not* closed under addition.

   (3) V is closed under scalar multiplication.

   ○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; y = |x| \right\}$ ○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; y \geq 2x \right\}$

   ● $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; xy \leq 0 \right\}$ ○ $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x^2 + y^2 \leq 1 \right\}$

   (b) (2 points) Suppose A is a 40×50 matrix and the solution set to $Ax = 0$ is 15-dimensional. Which **one** of the following describes the column space of A?

   ○ $\text{Col}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{50}$.

   ○ $\text{Col}(A)$ is a 15-dimensional subspace of $\mathbf{R}^{40}$.

   ● $\text{Col}(A)$ is a 35-dimensional subspace of $\mathbf{R}^{40}$.

   ○ $\text{Col}(A)$ is a 35-dimensional subspace of $\mathbf{R}^{50}$.

   ○ $\text{Col}(A)$ is a 25-dimensional subspace of $\mathbf{R}^{40}$.

   (c) (4 points) Suppose A is a square matrix with characteristic polynomial

   $$\det(A - \lambda I) = -\lambda(10 - \lambda)^2.$$

   Which of the following statements are true? Fill in the bubble for all that apply.

   ○ A cannot be diagonalizable.

   ● The null space of A must be a line.

   ○ Let B be the standard matrix for orthogonal projection onto the column space of A. Then the eigenvalues of B are -1 and 1.

   ● A is a 3×3 matrix.

   (d) (2 points) Find the value of c so that $\begin{pmatrix} -2 \\ 4 \\ c \end{pmatrix}$ and $\begin{pmatrix} c \\ -5 \\ -1 \end{pmatrix}$ are orthogonal. Clearly fill in the bubble for your answer.

   ○ $c = \frac{20}{3}$ ● $c = -\frac{20}{3}$ ○ $c = 0$ ○ $c = 20$

   ○ $c = -20$ ○ There is no value of c that makes them orthogonal.

Solution: Problem 2

(a) Set (i) is not closed under scalar multiplication because it contains $(1,1)$ but not $(-1,-1)$, as the line $y = |x|$ contains $(1,1)$ but not $(-1,-1)$.

Set (ii) is closed under addition: the set of vectors satisfying $y \geq 2x$ is all vectors on and above the line $y = 2x$, which is closed under addition and is not closed under scalar multiplication.

Set (iii) is the correct answer. First note it contains the zero vector. It is not closed under addition because for example $(1,0)$ and $(0,1)$ are both in the set ($xy = 0$ for both vectors), but their sum is $(1,1)$ which satisfies $xy > 0$. However, it is closed under scalar multiplication because if we take (x,y) with $xy \leq 0$ and scale it to get (cx, cy) then $(cx)(cy) = c^2 xy \geq 0$.

Set (iv) is not closed under scalar multiplication, since for example it contains (10) but not (50) since $5^2 + 0^2 > 1$.

(b) A is 40×50 so its column space is a subspace of $\mathbf{R}^{40}$. Also, by the Rank Theorem, we have

$$\dim(\text{Col } A) + \dim(\text{Nul } A) = 50, \quad \dim(\text{Col } A) + 15 = 50, \quad \dim(\text{Col } A) = 35.$$

(c) Statement (i) is not true, for example A could be $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix}$.

Statement (ii) is true: the eigenvalue $\lambda = 0$ has algebraic multiplicity 1, therefore its geometric multiplicity must also be 1, so the null space of A (i.e. the 0-eigenspace) is a line.

Statement (iii) is not true: $\lambda = -1$ is never an eigenvalue of a matrix for orthogonal projection.

Statement (iv) is true: its char. poly. has degree 3, so A must be 3×3.

(d) The dot product of the two vectors is $-2c - 20 - c$, and setting this to 0 gives $c = -20/3$.

3. Multiple choice. There is no partial credit on this page, and you do not need to show your work.

(a) (2 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that first reflects vectors across the line $y = -x$, then rotates vectors by $90°$ clockwise. Find $T\begin{pmatrix}4\\0\end{pmatrix}$. Fill in the bubble for your answer below.

○ $\begin{pmatrix}4\\0\end{pmatrix}$ ○ $\begin{pmatrix}0\\-4\end{pmatrix}$ ● $\begin{pmatrix}-4\\0\end{pmatrix}$ ○ $\begin{pmatrix}-2\\2\end{pmatrix}$ ○ $\begin{pmatrix}2\\-2\end{pmatrix}$

(b) (2 points) Suppose A is a matrix with columns $v_1, \ldots, v_k$. Which **one** of the following is **NOT** true? Clearly fill in the bubble for your one answer.

○ $\dim(\text{Row } A) = \dim(\text{Col } A)$.

○ $(\text{Row } A)^\perp = \text{Nul}(A)$.

○ $\text{Col}(A) = (\text{Nul } A^T)^\perp$.

● For the subspace $W = \text{Span}\{v_1, \ldots, v_k\}$, we have $W^\perp = \text{Nul}(A)$.

(c) (3 points) Select the **one** matrix A below that satisfies both of the following conditions:

- The only eigenvalue of A is $\lambda = 4$.
- The 4-eigenspace of A is a line.

Fill in the bubble for your answer below.

○ $A = \begin{pmatrix}4 & 1\\1 & 4\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & 1 & 0\\0 & 4 & 0\\0 & 0 & 4\end{pmatrix}$ ● $A = \begin{pmatrix}4 & -1 & 1 & 0\\0 & 4 & 1 & 0\\0 & 0 & 4 & 2\\0 & 0 & 0 & 4\end{pmatrix}$

○ $A = \begin{pmatrix}4 & 1 & -1 & 2\\0 & 4 & 1 & 0\\0 & 0 & 4 & 0\\0 & 0 & 0 & 4\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & -2 & 0\\0 & 4 & 0\\0 & 0 & 1\end{pmatrix}$ ○ $A = \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$

(d) (3 points) Let W be the subspace of $\mathbf{R}^4$ consisting of all vectors $\begin{pmatrix}x\\y\\z\\w\end{pmatrix}$ that satisfy $x - 2y - 3z + w = 0$, and let B be the standard matrix for orthogonal projection onto W. Which of the following are true? Fill in the bubble for all that apply.

○ $\dim(\text{Nul}(B)) = 3$ ○ $B\begin{pmatrix}1\\-2\\-3\\1\end{pmatrix} = \begin{pmatrix}1\\-2\\-3\\1\end{pmatrix}$ ● $Bx = x$ for all x in W

Solution: Problem 3

(a) Reflecting $\begin{pmatrix}4\\0\end{pmatrix}$ across the line $y = -x$ gives us $\begin{pmatrix}0\\-4\end{pmatrix}$, then rotating this vector by 90° clockwise to get $\begin{pmatrix}-4\\0\end{pmatrix}$ as our final answer.

Alternatively, we could have computed the matrix for T. It is AB where A is the second operation and B is the first operation. This means $A = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}$ and $B = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$, so

$$T\begin{pmatrix}0\\4\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}\begin{pmatrix}4\\0\end{pmatrix} = \begin{pmatrix}0 & 1\\-1 & 0\end{pmatrix}\begin{pmatrix}0\\-4\end{pmatrix} = \begin{pmatrix}-4\\0\end{pmatrix}.$$

(b) All are standard formulas from section 6.2 except for one. It is not correct that $W^\perp = \mathrm{Nul}(A)$. Instead, the correct formula is $W^\perp = \mathrm{Nul}(A^T)$.

(c) $\begin{pmatrix}4 & 1\\1 & 4\end{pmatrix}$ doesn't have 4 as an eigenvalue, $\begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$ has all of $\mathbf{R}^2$ as its eigenspace, and $\begin{pmatrix}4 & -2 & 0\\0 & 4 & 0\\0 & 0 & 1\end{pmatrix}$ also has 1 as an eigenvalue, so all three are clearly wrong.

$\begin{pmatrix}4 & 1 & 0\\0 & 4 & 0\\0 & 0 & 4\end{pmatrix}$ has a plane as its 4-eigenspace since $\left(A - 4I \mid 0\right) = \left(\begin{array}{ccc|c}0 & 1 & 0 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{array}\right)$.

$\left(\begin{array}{ccc|c}4 & 1 & -1 & 2\\0 & 4 & 1 & 0\\0 & 0 & 4 & 0\\0 & 0 & 0 & 4\end{array}\right)$ has a plane as its 4-eig. since $\left(A - 4I \mid 0\right) = \left(\begin{array}{cccc|c}0 & 1 & -1 & 2 & 0\\0 & 0 & 1 & 0 & 0\\0 & 0 & 0 & 0 & 0\\0 & 0 & 0 & 0 & 0\end{array}\right)$ which has two free variables.

However, the matrix $\begin{pmatrix}4 & -1 & 1 & 0\\0 & 4 & 1 & 0\\0 & 0 & 4 & 2\\0 & 0 & 0 & 4\end{pmatrix}$ has $\lambda = 4$ as its only eigenvalue, and the 4-eigenspace is a line since $\left(A - 4I \mid 0\right)$ has three pivots and thus 1 free variable in the solution set.

(d) (i) is false: $\dim(W) = 3$ and $\mathrm{Nul}(B) = W^\perp$ which satisfies $\dim(W^\perp) = 1$.
(ii) is false: we see $(1, -2, -3, 1)$ is not in W since $1 - 2(-2) - 3(-3) + 1 \neq 0$, so $Bx \neq x$.
(iii) is true: this is a standard fact about orthogonal projections.

4. Multiple choice. You do not need to show your work, and there is no partial credit.

(a) (3 points) Suppose $\{v_1, v_2, v_3, v_4\}$ is a linearly independent set of vectors in $\mathbf{R}^n$. Which of the following statements are true? Fill in the bubble for all that apply.

○ If b is a vector in $\mathbf{R}^n$, then the equation $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4 = b$ must have exactly one solution.

● The set $\{v_1, v_2, v_3\}$ is linearly independent.

● Suppose A is the matrix whose four columns are v_1, v_2, v_3, and v_4. Then the linear transformation $T(x) = Ax$ must be one-to-one.

(b) (2 points) Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x - 3y + z = 0 \right\}$. Which **one** of the following is equal to W? Fill in the bubble for your answer.

○ $\text{Nul}\begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ ● $\text{Col}\begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ ○ $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}\right\}$ ○ $\text{Nul}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$

(c) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} 3a-d & 3b-e & 3c-f \\ a & b & c \\ 5g & 5h & 5i \end{pmatrix}$.

○ 1 ○ -1 ○ 3 ○ -3 ● 5 ○ -5

○ 10 ○ -10 ○ 15 ○ -15 ○ none of these

(d) (3 points) Abacus Seltzer and Barren Seltzer compete for a market of 100 customers who drink seltzer each day. Today, Abacus Seltzer has 25 customers and Barren Seltzer has 75 customers. Each day:

- 80% of Abacus Seltzer's customers keep drinking Abacus Seltzer, while 20% switch to Barren Seltzer.
- 65% of Barren Seltzer's customers keep drinking Barren Seltzer, while 35% switch to Abacus Seltzer.

Fill in the bubble below that gives a positive stochastic matrix A and a vector x so that Ax will give the number of customers for Abacus Seltzer and Barren Seltzer (in that order) tomorrow.

○ $A = \begin{pmatrix} 0.8 & 0.2 \\ 0.65 & 0.35 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$ ○ $A = \begin{pmatrix} 0.8 & 0.65 \\ 0.2 & 0.35 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$

● $A = \begin{pmatrix} 0.8 & 0.35 \\ 0.2 & 0.65 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$ ○ $A = \begin{pmatrix} 0.8 & 0.2 \\ 0.35 & 0.65 \end{pmatrix}$, $x = \begin{pmatrix} 25 \\ 75 \end{pmatrix}$

Solution: Problem 4

(a) Statement (i) is not necessarily true: if b is a vector in $\mathbf{R}^n$ but b is not in W, then $x_1v_1 + x_2v_2 + x_3v_3 + x_4v_4$ has no solution.
Statement (ii) is true: $\{v_1, v_2, v_3\}$ is a subset of the linearly independent set $\{v_1, v_2, v_3, v_4\}$, so it must be linearly independent.
Statement (iii) is true, and it is a standard fact. If the columns of a matrix A are linearly independent, then its corresponding matrix transformation $T(x) = Ax$ is one-to-one.

(b) $W = \text{Nul}\begin{pmatrix} 1 & -3 & 1 \end{pmatrix}$, so we quickly find that $\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ is a basis for W, therefore that $W = \text{Col}\begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$.
All other answers are wrong: note that $\text{Nul}\begin{pmatrix} 3 & -1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ is just the zero vector, $\text{Span}\left\{ \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \right\}$ is just a line inside of W, and $\text{Nul}\begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix}$ is the zero vector.

(c) The original det is 1. If we swap the first two rows (multiplying det by -1) and then multiply the new first row by -1 (multiplying det by -1 again), we get the new matrix $\begin{pmatrix} -d & -e & -f \\ a & b & c \\ g & h & i \end{pmatrix}$ which has determinant $1(-1)(-1) = 1$.
From here, we add $3R_2$ to R_1 which doesn't change the determinant, then we multiply the final row by 5 to get a final determinant of 5. Therefore,

$$\det \begin{pmatrix} 3a - d & 3b - e & 3c - f \\ a & b & c \\ 5g & 5h & 5i \end{pmatrix} = 5.$$

(d) This is a standard stochastic matrix problem, but can be done without even using the theory of that section. From the text, tomorrow's Abacus customers are 80% of its correct customers plus 35% of Barren's customers, which is

$$0.8(25) + 0.35(75).$$

Tomorrow's Barren customers are 65% of its current customers plus 20% of Abacus's customers, which is

$$0.2(25) + 0.65(75).$$

This corresponds to $\begin{pmatrix} 0.8 & 0.35 \\ 0.2 & 0.65 \end{pmatrix} \begin{pmatrix} 25 \\ 75 \end{pmatrix}$.

5. Multiple choice. You do not need to show your work on this page, and there is no partial credit.

(a) (2 points) Which **one** of the following matrices A has the property that $\mathrm{Col}(A) = \mathrm{Nul}(A)$? Clearly fill in the bubble for your answer.

○ $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ ● $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

○ $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 1 & -1 \end{pmatrix}$ ○ $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ ○ $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(b) (2 points) Find the area of the triangle with vertices

$$(1, 2), \qquad (5, -1), \qquad (6, -4).$$

Fill in the bubble for your answer below.

○ 5 ○ 9 ○ 11 ○ 15 ○ 19 ○ 39

● $\frac{9}{2}$ ○ $\frac{11}{2}$ ○ $\frac{19}{2}$ ○ $\frac{39}{2}$ ○ none of these

(c) (3 points) Suppose W is a subspace of $\mathbf{R}^n$. Which of the following statements are true? Fill in the bubble for all that apply.

● If x is a vector in $W^\perp$, then the orthogonal projection of x onto W must be the zero vector.

○ If x is a vector in $\mathbf{R}^n$, then x must be in W or $W^\perp$.

● If x is a vector in both W and $W^\perp$, then x must be the zero vector.

(d) (3 points) Suppose A is an $m \times n$ matrix and b is a vector in $\mathbf{R}^m$. Which of the following statements are true? Fill in the bubble for all that apply.

● The equation $Ax = b$ is consistent if and only if b is a linear combination of the columns of A.

● If w is the orthogonal projection of b onto the column space of A, then the equation $Ax = w$ must be consistent.

○ If $\widehat{x}$ is a least-squares solution to $Ax = b$, then $\widehat{x}$ is the closest vector to b in $\mathrm{Col}(A)$.

Solution: Problem 5

(a) It is never possible for a 3×3 matrix to have $\mathrm{Col}(A) = \mathrm{Nul}(A)$ because that would make each of them "1.5-dimensional" which is nonsense. This means we can rule out all the 3×3 matrices here.

- $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ has column space spanned by $(1, 1)$ but null space spanned by $(1, -1)$, so its column space and null space are different.
- $\begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ has column space spanned by $(1, 0)$ but null space spanned by $\begin{pmatrix} 1 & 1 \end{pmatrix}$, so its column space and null space are different.

So $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Here $\mathrm{Col}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$, and $\left(A \mid 0 \right) = \left(\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right)$ so its null space is given by x_1 free with $x_2 = 0$, thus $\mathrm{Nul}(A) = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$.

(b) The vector from $(1, 2)$ to $(5, -1)$ is $v_1 = \begin{pmatrix} 4 \\ -3 \end{pmatrix}$.

The vector from $(1, 2)$ to $(6, -4)$ is $v_2 = \begin{pmatrix} 5 \\ -6 \end{pmatrix}$. The area of the triangle is

$$\frac{1}{2}\left|\det\begin{pmatrix} v_1 & v_2 \end{pmatrix}\right| = \frac{1}{2}\left|\det\begin{pmatrix} 4 & 5 \\ -3 & -6 \end{pmatrix}\right| = \frac{1}{2}\left|-24 - (-15)\right| = \frac{9}{2}.$$

(c) Statement (i) is true: if x is in $W^\perp$ then $x = x_{W^\perp}$ and $x_W = 0$.

Statement (ii) is not true: by Orthogonal Decomposition, every x in $\mathbf{R}^n$ can be written as a sum of a vector in W and a vector in $W^\perp$, as $x = x_W + x_{W^\perp}$. This does NOT mean that x is necessarily in W or $W^\perp$. For example, take $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $W = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$. Then $W^\perp = \mathrm{Span}\left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$, and we see x is not in W or $W^\perp$, but $x = x_W + x_{W^\perp}$ where $x_W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $x_{W^\perp} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Statement (iii) is true: if x is in both W and $W^\perp$, then x is orthogonal to itself. In other words, $x \perp x$, which means $||x||^2 = 0$. The only vector with length 0 is the zero vector.

(d) Statement (i) is true. It is a fundamental fact from chapter 2.
Statement (ii) is true: w is in the column space of A since $w = b_{\mathrm{Col}(A)}$, therefore $Ax = w$ must be consistent.
Statement (iii) is false: $A\widehat{x}$ is the closest vector to b in $\mathrm{Col}(A)$.

6. Multiple choice and short answer. You do not need to show your work on this page, and there is no partial credit except on (d).

(a) (2 points) Suppose $T : \mathbf{R}^k \to \mathbf{R}^\ell$ is a transformation. Which **one** of the following conditions guarantees that T is one-to-one? Fill in the bubble for your answer.

○ For each b in $\mathbf{R}^\ell$, there is at least one a in $\mathbf{R}^k$ so that $T(a) = b$.

● For each b in $\mathbf{R}^\ell$, there is at most one a in $\mathbf{R}^k$ so that $T(a) = b$.

○ For each a in $\mathbf{R}^k$, there is at least one b in $\mathbf{R}^\ell$ so that $T(a) = b$.

○ For each a in $\mathbf{R}^k$, there is at most one b in $\mathbf{R}^\ell$ so that $T(a) = b$.

(b) (2 points) Let A be the matrix that reflects every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = -10x$. Which **one** of the following vectors is an eigenvector corresponding to the eigenvalue $\lambda = -1$? Fill in the bubble for your answer.

○ $\begin{pmatrix} 1 \\ -10 \end{pmatrix}$ ○ $\begin{pmatrix} -10 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 10 \end{pmatrix}$ ● $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ 5 \end{pmatrix}$

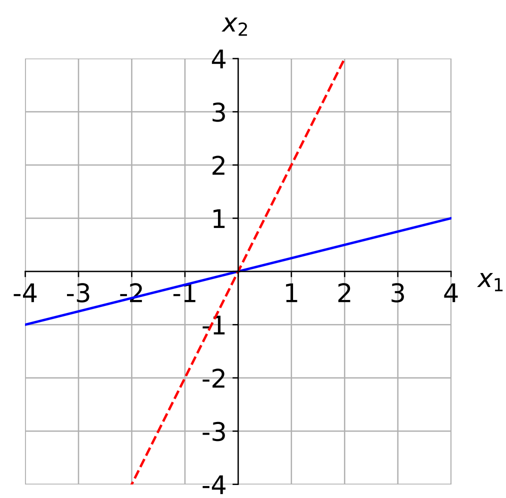
(c) (2 points) Suppose A is a 3×3 matrix whose 4-eigenspace is a plane and whose (-2)-eigenspace is a line. Find $\det(A)$. Fill in the bubble for your answer below.

○ -2 ○ 4 ○ 8 ○ -8 ○ 16

○ -16 ○ 32 ● -32 ○ not enough info to compute $\det(A)$

(d) (4 points) Write a single matrix A whose column space is the **solid** line below and whose null space is the **dashed** line below.

Many answers possible, for example
$A = \begin{pmatrix} 4 & -2 \\ 1 & -1/2 \end{pmatrix}$.



Solution: Problem 6

(a) The correct answer is "For each b in $\mathbf{R}^\ell$, there is at most one a in $\mathbf{R}^k$ so that $T(a) = b$." This is almost word-for-word the definition of one-to-one.

Of the others: the two options that begin with "For each a in $\mathbf{R}^k$" are modifications of the definition of transformation, while the other "For each b in $\mathbf{R}^\ell$" option is nearly word-for-word the definition of onto.

(b) The line of reflection is $y = -10x$, so the (-1)-eigenspace is the perpendicular line through the origin, which is the line $y = x/10$. The one answer choice on this line is $\begin{pmatrix} 10 \\ 1 \end{pmatrix}$, which we can check is perpendicular to the line $y = -10x$ since

$$\begin{pmatrix} 10 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -10 \end{pmatrix} = 10 - 10 = 0.$$

(c) From the information given, it must be the case that

$$\det(A - \lambda I) = (4 - \lambda)^2(-2 - \lambda).$$

Therefore, $\det(A) = \det(A - 0I) = 4^2(-2) = -32$.
Another way to see this is to note that since A is 3×3 with three linearly independent eigenvectors, it is diagonalizable, so $A = CDC^{-1}$ where $D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -2 \end{pmatrix}$. Note

$\det(A) = \det(CDC^{-1}) = \cancel{\det(C)}\det(D)\cancel{\det(C^{-1})} = \det(D) = 4(4)(-2) = -32.$

(d) We need $\operatorname{Col}(A) = \operatorname{Span}\left\{\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right\}$, so each column is a scalar multiple of $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.
We also need $\operatorname{Nul}(A) = \operatorname{Span}\left\{\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right\}$ which means $x_1 = \frac{x_2}{2}$, therefore $x_1 - \frac{1}{2}x_2 = 0$. This means the second column must be $-1/2$ times the first column. Possible answers for A are

$$A = \begin{pmatrix} 4 & -2 \\ 1 & -1/2 \end{pmatrix}, \qquad A = \begin{pmatrix} -4 & 2 \\ -1 & 1/2 \end{pmatrix}, \qquad \text{etc.}$$

7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may result in little or no credit.

Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that rotates vectors in $\mathbf{R}^2$ by $90°$ **counterclockwise**, and let $U : \mathbf{R}^3 \to \mathbf{R}^2$ be the linear transformation given by

$$U(x, y, z) = (-3x + y - 2z,\ x + 4y - 5z).$$

(a) (2 points) Find the standard matrix A for T. Enter your answer below. Evaluate any trigonometric functions. Do not leave your answer in terms of sine and cosine.

$$A = \begin{pmatrix} \cos(90°) & -\sin(90°) \\ \sin(90°) & \cos(90°) \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(b) (3 points) Is there a matrix B so that AB is the 2×2 identity matrix? If your answer is yes, find B and write it in the space below. If your answer is no, fill in the bubble for "no such B exists" and justify your answer.

This is just asking if A is invertible, and if so to find A^{-1}. The answer is yes, and B is the matrix for rotation $90°$ clockwise. We could make this observation geometrically without any formulas, or we could compute $B = A^{-1}$.

$$B = A^{-1} = \frac{1}{0 - (-1)} \begin{pmatrix} 0 & -(-1) \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

(c) (3 points) Find the standard matrix C for U. Enter your answer below.

$$C = \left(U\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad U\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad U\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} -3 & 1 & -2 \\ 1 & 4 & -5 \end{pmatrix}$$

(d) (2 points) Find the standard matrix M for $T \circ U$. Enter your answer below.

The matrix is $AC = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} -3 & 1 & -2 \\ 1 & 4 & -5 \end{pmatrix} = \begin{pmatrix} -1 & -4 & 5 \\ -3 & 1 & -2 \end{pmatrix}.$

8. Free response. This page is worth 10 points. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Consider the matrix $A = \begin{pmatrix} 4 & 1 & -1 \\ 0 & 0 & 4 \\ 0 & 1 & 3 \end{pmatrix}$.

(a) Find the eigenvalues of A. Enter them here: ______________________.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(c) A is diagonalizable. In the space provided below, write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. You do not need to show your work on this part.

Solution:

(a) Cofactor expansion along the first column gives

$$\det(A - \lambda I) = \det \begin{pmatrix} 4-\lambda & 1 & -1 \\ 0 & -\lambda & 4 \\ 0 & 1 & 3-\lambda \end{pmatrix} = (4-\lambda)\Big[(-\lambda)(3-\lambda) - 4\Big]$$
$$= (4-\lambda)\Big[\lambda^2 - 3\lambda - 4\Big] = (4-\lambda)(\lambda-4)(\lambda+1).$$

The eigenvalues are $\lambda_1 = 4$ and $\lambda_2 = -1$.

(b)

$$\lambda = -1 : \big(A + I \,|\, 0\big) = \left(\begin{array}{ccc|c} 5 & 1 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right) \xrightarrow[R_1 = R_1 - R_2,\text{ then } R_1 = \frac{R_1}{5}]{R_3 = R_3 - R_2} \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So $x_1 = x_3$, $x_2 = -4x_3$, and x_3 is free.

A basis for the (-1)-eig. is $\left\{ \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \right\}$.

$$\lambda = 4 : \ \big(A - 4I \,|\, 0\big) = \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right).$$

So x_1 and x_3 are free and $x_2 = x_3$. A basis for the 4-eigenspace is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \right\}$.

(c) We've found 3 linearly independent eigenvectors, so A is diagonalizable: $A = CDC^{-1}$ where

$$C = \begin{pmatrix} 1 & 1 & 0 \\ -4 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

9. Free response. Show your work! A correct answer without sufficient work may result in little or no credit.

   (a) (3 points) Let $W = \text{Span}\left\{\begin{pmatrix}1\\0\\7\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$. Find a basis for $W^\perp$.

   $W^\perp = \text{Nul}\begin{pmatrix}1 & 0 & 7\\ 0 & 1 & 0\end{pmatrix}$, which gives us $x_1 + 7x_3 = 0$ and $x_2 = 0$ with x_3 free.

   This gives us $\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} = \begin{pmatrix}-7x_3\\0\\x_3\end{pmatrix}$. Therefore, a basis for $W^\perp$ is Span $\left\{\begin{pmatrix}-7\\0\\1\end{pmatrix}\right\}$.

   (b) Let $W = \text{Span}\left\{\begin{pmatrix}-1\\2\end{pmatrix}\right\}$ and let $y = \begin{pmatrix}5\\15\end{pmatrix}$.

   i. (4 pts) Find the closest vector to y in W. Enter your answer here: $\begin{pmatrix}-5\\10\end{pmatrix}$

   Solution: Many methods possible. With $A = u = \begin{pmatrix}-1\\2\end{pmatrix}$, we could solve $A^TAv = A^Ty$ and this would give us $5v = 25$ so $v = 5$, therefore

   $$y_W = Av = \begin{pmatrix}-1\\2\end{pmatrix} * 5 = \begin{pmatrix}-5\\10\end{pmatrix}.$$

   Alternatively, we could have computed $y_W = By$ where

   $$B = \frac{1}{u \cdot u}uu^T = \frac{1}{(-1)^2 + 2^2}\begin{pmatrix}-1\\2\end{pmatrix}\begin{pmatrix}-1 & 2\end{pmatrix} = \frac{1}{5}\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}.$$

   So $y_W = Bw = \frac{1}{5}\begin{pmatrix}1 & -2\\ -2 & 4\end{pmatrix}\begin{pmatrix}5\\15\end{pmatrix} = \frac{1}{5}\begin{pmatrix}-25\\50\end{pmatrix} = \begin{pmatrix}-5\\10\end{pmatrix}$.
   Another method would be to use the formula

   $$y_W = \frac{u \cdot y}{u \cdot u}u = \frac{\begin{pmatrix}-1\\2\end{pmatrix} \cdot \begin{pmatrix}5\\15\end{pmatrix}}{(-1)^2 + 2^2}\begin{pmatrix}-1\\2\end{pmatrix} = \frac{25}{5}\begin{pmatrix}-1\\2\end{pmatrix} = \begin{pmatrix}-5\\10\end{pmatrix}.$$

   ii. (3 points) Find the distance from y to W. Enter your answer $\sqrt{125}$ (or $5\sqrt{5}$).
   Solution: The distance from y to W is $||y_W^\perp||$.
   We find $y_{W^\perp} = y - y_W = \begin{pmatrix}5\\15\end{pmatrix} - \begin{pmatrix}-5\\10\end{pmatrix} = \begin{pmatrix}10\\5\end{pmatrix}$, and

   $$||y_{W^\perp}|| = \sqrt{10^2 + 5^2} = \sqrt{100 + 25} = \sqrt{125} = 5\sqrt{5}.$$

10. Free response. This problem is worth 10 points. Show your work!

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(1, 10), \qquad (2, -8), \qquad (3, 4).$$

Enter your answer below:

$$y = \underline{\hspace{3cm}}x \ + \ \underline{\hspace{3cm}}.$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

No line goes through all three points. The corresponding (inconsistent) system is

$$10 = M(1) + B$$

$$-8 = M(2) + B$$

$$4 = M(3) + B$$

and the corresponding matrix equation is $Ax = b$ where $A = \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} 10 \\ -8 \\ 4 \end{pmatrix}$.

We solve $A^T A\widehat{x} = A^T b$.

$$A^T A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 14 & 6 \\ 6 & 3 \end{pmatrix}, \qquad A^T b = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 10 \\ -8 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \end{pmatrix}.$$

$$\left(A^T A \mid A^T b \right) = \left(\begin{array}{cc|c} 14 & 6 & 6 \\ 6 & 3 & 6 \end{array}\right) \xrightarrow[\text{then } R_1 = R_1/3]{R_1 \leftrightarrow R_2} \left(\begin{array}{cc|c} 2 & 1 & 2 \\ 14 & 6 & 6 \end{array}\right) \xrightarrow{R_2 = R_2 - 7R_1} \left(\begin{array}{cc|c} 2 & 1 & 2 \\ 0 & -1 & -8 \end{array}\right)$$

$$\xrightarrow[\text{then } R_1 = R_1 - R_2]{R_2 = -R_2} \left(\begin{array}{cc|c} 2 & 0 & -6 \\ 0 & 1 & 9 \end{array}\right) \xrightarrow{R_1 = R_1/2} \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 8 \end{array}\right).$$

Thus $\widehat{x} = \begin{pmatrix} -3 \\ 8 \end{pmatrix}$. The line is

$$y = -3x + 8.$$

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If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.