You can find many past final exams at Prof. Jankowski's exam archive, which is linked at his Georgia Tech website. This practice exam is one version of a past final exam. It is not meant to foreshadow or predict final exam problems this semester. Instead, it is simply one of many practice exams available to you.

Math 1553 Practice Final, Fall 2024

Name		**GT ID**	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15) Wessels(B, 8:25-9:15) Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20) Kim (G, 12:30-1:20) Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50) He (L, 3:30-4:20) Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50) Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 100 points, and you have 170 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until 8:50 PM on Tuesday, December 10.

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1. (1 pt each) TRUE or FALSE. If the statement is **ever** false, fill in the bubble for false. You do not need to show work or justify your answer. As stated in the instructions, **the entries of all matrices on the exam are real numbers** unless stated otherwise.

(a) Suppose $\{v_1, v_2, v_3, v_4, v_5\}$ is a basis for $\mathbf{R}^n$. Then $n = 5$.

○ True ○ False

(b) Suppose $T : \mathbf{R}^{20} \to \mathbf{R}^7$ is a linear transformation with standard matrix A, so $T(x) = Ax$. Then $\dim(\text{Nul } A) \geq 13$.

○ True ○ False

(c) Let A be an $m \times n$ matrix, and let T be the corresponding matrix transformation $T(x) = Ax$. If $m > n$, then T cannot be onto.

○ True ○ False

(d) The set W of all vectors (x, y, z) in $\mathbf{R}^3$ with $x - y + z = 3$ is a subspace of $\mathbf{R}^3$.

○ True ○ False

(e) There is a 3×3 matrix A, whose entries are real numbers, so that $2 - i$ and $3i$ are eigenvalues of A.

○ True ○ False

(f) Every nonzero vector in $\mathbf{R}^3$ is an eigenvector of the 3×3 identity matrix.

○ True ○ False

(g) Suppose that u and v are vectors in the 4-eigenspace of some $n \times n$ matrix A. Then $4u - 3v$ must also be in the 4-eigenspace of A.

○ True ○ False

(h) Let A be a 3×3 matrix satisfying $Ae_1 = 3e_1$, $Ae_2 = 3e_2$, and $\det(A - \lambda I) = (1 - \lambda)(3 - \lambda)^2$. Then A must be diagonalizable.

○ True ○ False

(i) Suppose A is an $m \times n$ matrix and b is a vector in the column space of A. Then every solution to $Ax = b$ is also a least squares solution to $Ax = b$.

○ True ○ False

(j) Suppose that W is a subspace of $\mathbf{R}^n$ and that u is a vector in W. Then the orthogonal projection of u onto W is the zero vector.

○ True ○ False

2. Short answer. You do not need to show your work on this page, and (a)-(d) are unrelated.

(a) (3 points) Let $V = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \text{ in } \mathbf{R}^2 \;\middle|\; x^2 + y^2 \leq 5 \right\}$. Answer the following questions.

(i) Does V contain the zero vector? ○ Yes ○ No

(ii) Is V closed under addition? In other words, if u and v are in V, must it be true that $u + v$ is in V? ○ Yes ○ No

(iii) Is V closed under scalar multiplication? In other words, if c is a real number and u is in V, must it be true that cu is in V? ○ Yes ○ No

(b) (3 points) Suppose that A is a 2024×100 matrix.
Which of the following are **possible**? Clearly fill in the bubble for all that apply.

○ The dimension of Row(A) is 105.

○ The dimension of Nul(A) is 105.

○ The transformation $T(x) = Ax$ is one-to-one.

(c) (2 points) Let A be a 35×50 matrix that has 20 pivots. Which one of the following describes the null space of A? Clearly fill in the bubble for your answer.

○ Nul(A) is a 30-dimensional subspace of $\mathbf{R}^{35}$.

○ Nul(A) is a 20-dimensional subspace of $\mathbf{R}^{50}$.

○ Nul(A) is a 15-dimensional subspace of $\mathbf{R}^{35}$.

○ Nul(A) is a 30-dimensional subspace of $\mathbf{R}^{50}$.

(d) (2 pts) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -1 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 3 \end{pmatrix} \right\}$, and let $v = \begin{pmatrix} 2 \\ 1 \\ 1 \\ -1 \end{pmatrix}$.

Which of the following statements are true? Clearly fill in the bubble for all that apply.

○ The set $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}$ is a basis for W.

○ v is in $W^{\perp}$.

3. Short answer. Parts (a)-(c) are unrelated. You do not need to show your work on this page, and there is no partial credit except on part (c).

(a) (4 points) Suppose $T : \mathbf{R}^a \to \mathbf{R}^b$ is a linear transformation. Which of the following conditions guarantee that T is onto? Clearly fill in the bubble for **all** that apply.

○ For each y in $\mathbf{R}^b$, there is at least one x in $\mathbf{R}^a$ so that $T(x) = y$.

○ For each x in $\mathbf{R}^a$, there is at most one y in $\mathbf{R}^b$ so that $T(x) = y$.

○ For each x in $\mathbf{R}^a$, there is exactly one y in $\mathbf{R}^b$ so that $T(x) = y$.

○ For each y in $\mathbf{R}^b$, there is exactly one x in $\mathbf{R}^a$ so that $T(x) = y$.

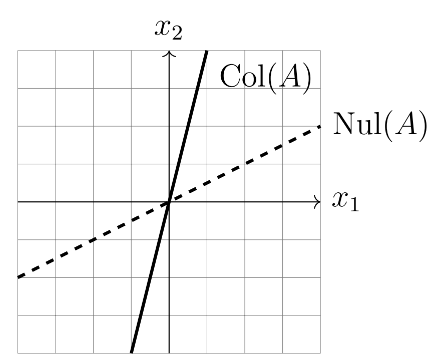
(b) (3 points) Which of the following matrices A are invertible? Clearly fill in the bubble for **all** that apply.

○ $A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & -2 & 2 \\ 0 & 1 & 1 \end{pmatrix}$

○ Any 3×3 matrix A that has eigenvalues $\lambda = 1$, $\lambda = -1$, and $\lambda = 3$.

○ The 2×2 matrix A that rotates vectors in $\mathbf{R}^2$ by 60 degrees counterclockwise.

(c) (3 points) Write a matrix A so that $\mathrm{Col}(A)$ is the **solid** line below and $\mathrm{Nul}(A)$ is the **dashed** line below.



Enter your answer here: $A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$

4. Short answer. Parts (a) through (d) are unrelated. You do not need to show your work on this page, and there is no partial credit.

(a) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 1$. Find $\det\begin{pmatrix} d & e & f \\ 4d-3a & 4e-3b & 4f-3c \\ g & h & i \end{pmatrix}$.

Clearly fill in the bubble for your answer below.

○ 1 ○ -1 ○ 3 ○ -3 ○ 4

○ -4 ○ 12 ○ -12 ○ -24 ○ none of these

(b) (2 points) Find the area of the triangle with vertices $(-4,4)$, $(2,-2)$, and $(6,-1)$.

○ $3/2$ ○ 3 ○ $15/2$ ○ 15 ○ 30

○ $45/2$ ○ 45 ○ 50 ○ 90 ○ none of these

(c) (2 points) Suppose A and B are 3×3 matrices satisfying $\det(A) = 4$ and $\det(B) = 2$. Find $\det(-3A^{-1}B)$.

○ $-3/2$ ○ $9/2$ ○ $-27/2$ ○ -24 ○ $27/2$

○ -72 ○ $-9/2$ ○ 72 ○ none of these

(d) (4 points) Suppose A is a 4×4 matrix with characteristic polynomial

$$\det(A - \lambda I) = (2-\lambda)^2(3-\lambda)(-1-\lambda).$$

Which of the following statements are true? Clearly fill in the bubble for all that apply.

○ A is invertible.

○ If the 2-eigenspace of A is a plane, then A must be diagonalizable.

○ It is possible that the 3-eigenspace of A is a plane.

○ The zero vector is **not** an eigenvector of A.

5. Short answer. You do not need to show your work on this page, and there is no partial credit. Parts (a)-(d) are unrelated.

   (a) (3 points) Let A be a 2×2 matrix whose entries are real numbers. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

   ○ If A has $\lambda = -5$ as an eigenvalue with algebraic multiplicity 2, then -5 is the only eigenvalue of A.

   ○ If $\lambda = 1 - 4i$ is an eigenvalue of A, then A does not have any real eigenvalues.

   ○ If A is a stochastic matrix, then the only eigenvalue of A is $\lambda = 1$.

   (b) (3 points) Let A be the 2×2 matrix that reflects vectors $\begin{pmatrix} x \\ y \end{pmatrix}$ across the line $y = 3x$. Which of the following are true? Fill in the bubble for all that apply.

   ○ The eigenvalues of A are $\lambda = 0$ and $\lambda = 1$.

   ○ A is diagonalizable.

   ○ $A\begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -3 \\ 1 \end{pmatrix}$.

   (c) (2 points) Let $A = \begin{pmatrix} 0.3 & 0.1 \\ 0.7 & 0.9 \end{pmatrix}$. It has the property that $A\begin{pmatrix} 1 \\ 7 \end{pmatrix} = \begin{pmatrix} 1 \\ 7 \end{pmatrix}$. What vector does $A^n \begin{pmatrix} 400 \\ 0 \end{pmatrix}$ approach as n gets large? Clearly fill in the bubble for your answer.

   ○ $\begin{pmatrix} 64 \\ 336 \end{pmatrix}$ ○ $\begin{pmatrix} 1/8 \\ 7/8 \end{pmatrix}$ ○ $\begin{pmatrix} 50 \\ 350 \end{pmatrix}$ ○ $\begin{pmatrix} 280 \\ 120 \end{pmatrix}$

   ○ $\begin{pmatrix} 120 \\ 280 \end{pmatrix}$ ○ $\begin{pmatrix} 280 \\ 120 \end{pmatrix}$ ○ $\begin{pmatrix} 50 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 400 \\ 0 \end{pmatrix}$

   (d) (2 points) Suppose W is a subspace of $\mathbf{R}^n$ and B is the matrix for orthogonal projection onto W. Which of the following must be true? Clearly fill in the bubble for all that apply.

   ○ The eigenvalues of B are $\lambda = -1$ and $\lambda = 1$.

   ○ $B^3 = B$.

6. Short answer. You do not need to show your work, and (a)-(c) are unrelated.

(a) (2 points) Find the value of c so that $\begin{pmatrix} 1 \\ -2 \\ c \\ 4 \end{pmatrix}$ and $\begin{pmatrix} 6 \\ c \\ 0 \\ -3 \end{pmatrix}$ are orthogonal.

Fill in the blank: $c =$ ______________.

(b) (5 points) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector so that

$$x_W = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}.$$

(i) What is the distance from x to W? Fill in the bubble for your answer.

○ $\sqrt{11}$ ○ $\sqrt{3}$ ○ $\sqrt{30}$ ○ $\sqrt{6}$ ○ $\sqrt{41}$

○ 11 ○ 3 ○ 30 ○ 41 ○ 51

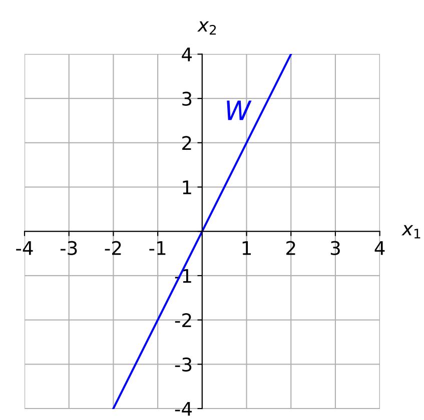
(ii) What is the closest vector to x in W? Fill in the bubble for your answer.

○ $\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} -6 \\ 1 \\ 2 \end{pmatrix}$ ○ $\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$ ○ $\begin{pmatrix} 4 \\ 5 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

(iii) Which **one** of the following **could** be $W^\perp$? Fill in the bubble for your answer.

○ Nul $\begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$ ○ Row $\begin{pmatrix} 5 & 2 & -1 \end{pmatrix}$ ○ Nul $\begin{pmatrix} -1 & 3 & 1 \end{pmatrix}$ ○ Col $\begin{pmatrix} -1 \\ 3 \\ 1 \end{pmatrix}$

(c) (3 points) Let W be the line graphed below, and let $x = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$. On the graph below, very carefully **draw** and **label** x, x_W, and $x_{W^\perp}$.



7. Free response. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

For this problem, let $A = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 3 & -15 & 13 \end{pmatrix}$.

(a) (2 pts) Write the eigenvalues of A. You do not need to show your work on this part.

Fill in the blank: the eigenvalues are ____________________.

(b) (6 points) For each eigenvalue of A, find a basis for the corresponding eigenspace.

(c) (2 points) The matrix A is diagonalizable. Write a 3×3 matrix C and a 3×3 diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix} \qquad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

8. Free response. Show your work! A correct answer without sufficient work may receive little or no credit.

   (a) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation corresponding to reflection over the line $y = x$. Find the standard matrix A for T, so $T(v) = Av$. Enter your answer below.

   $$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

   (b) Let $S : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation defined by $S(x, y, z) = (2x - y, y + 4z)$. Find the standard matrix B for S, so $S(v) = Bv$. Enter your answer below.

   $$B = \begin{pmatrix} \quad & \quad & \quad \\ \quad & \quad & \quad \end{pmatrix}$$

   (c) Which of the following expressions are possible to calculate? Clearly fill in the bubble for all that apply. You do not need to show your work on this part.

   ○ $T\begin{pmatrix} 3 \\ -5 \\ 8 \end{pmatrix}$ ○ $S\begin{pmatrix} 6 \\ 2 \\ -1 \end{pmatrix}$ ○ $(T \circ S)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ○ $(S \circ T)\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

   (d) Which one of the following compositions makes sense? Fill in the bubble for the **one** correct answer. You do not need to show your work on this part.

   ○ $T \circ S$ ○ $S \circ T$

   (e) Find the standard matrix C for the transformation you selected in part (d). Enter your answer below.

   $$C = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

9. Free response. Show your work! A correct answer without sufficient work may receive little or no credit. Parts (a) and (b) are unrelated.

   (a) Let $W = \text{Span}\left\{ \begin{pmatrix} -1 \\ 6 \\ -3 \end{pmatrix} \right\}$. Find a basis for $W^{\perp}$.

   (b) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ -4 \end{pmatrix} \right\}$ and let $x = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
   Find x_W (the orthogonal projection of x onto W) and $x_{W^\perp}$. Enter your answers below. **Simplify all fractions in your answer as much as possible**.

   $$x_W = \begin{pmatrix} \quad \\ \quad \end{pmatrix} \qquad\qquad x_{W^\perp} = \begin{pmatrix} \quad \\ \quad \end{pmatrix}.$$

10. Free response. Show your work!
Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0,5), \qquad (2,-5), \qquad (4,-3).$$

Enter your answer below:

$$y = \_\_\_\_\_\_ x \; + \; \_\_\_\_\_\_ .$$

You **must** show appropriate work using least squares. If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.