Math 1553: Some Additional Final Exam Practice Problems

Fall 2024

These problems are for extra practice for the final. They are not meant to be 100% comprehensive in scope.

1. Define the following terms: span, linear combination, linearly independent, linear transformation, column space, null space, transpose, inverse, dimension, rank, eigenvalue, eigenvector, eigenspace, diagonalizable, orthogonal.

2. Let A be an $m \times n$ matrix.

a) How do you determine the pivot columns of A?

b) What do the pivot columns tell you about the equation $Ax = b$?

c) What space is equal to the span of the pivot columns?

d) What is the difference between solving $Ax = b$ and $Ax = 0$? How are the two solution sets related geometrically?

e) If $\operatorname{rank}(A) = r$, where $0 \leq r \leq n$, then how many columns have pivots? What is the dimension of the null space?

Solution.

a) Do row operations until A is in a row echelon form. The leading entries of the rows are the pivots.

b) If there is a pivot in every column, then $Ax = b$ has zero or one solution. Otherwise, $Ax = b$ has zero or infinitely many solutions.

c) The pivot columns form a basis for the column space $\operatorname{Col} A$.

d) Suppose that $Ax = b$ has some solution x_0. Then every other solution to $Ax = b$ has the form $x_0 + x$, where x is a solution to $Ax = 0$. In other words, the solution set to $Ax = b$ is either empty, or it is a translate of the solution set to $Ax = 0$ (the null space).

e) If $\operatorname{rank}(A) = r$ then there are r pivot columns. The null space has dimension $n - r$.

3. Let $T : \mathbf{R}^n \to \mathbf{R}^m$ be a linear transformation with matrix A.

a) How many rows and columns does A have?

b) If x is in $\mathbf{R}^n$, then how do you find $T(x)$?

c) In terms of A, how do you know if T is one-to-one? onto?

d) What is the range of T?

Solution.

a) A has m rows and n columns.

b) $T(x) = Ax$.

c) T is one-to-one if and only if A has a pivot in every column. T is onto if and only if A has a pivot in every row.

d) $\mathrm{Col}\,A$.

4. Let A be an invertible $n \times n$ matrix.

a) What can you say about the columns of A?

b) What are rank(A) and $\dim \mathrm{Nul}\,A$?

c) What do you know about det(A)?

d) How many solutions are there to $Ax = b$? What are they?

e) What is $\mathrm{Nul}\,A$?

f) Do you know anything about the eigenvalues of A?

g) Do you know whether or not A is diagonalizable?

Solution.

a) The columns are linearly independent. They also span $\mathbf{R}^n$.

b) $\mathrm{rank}(A) = n$ and $\dim \mathrm{Nul}\,A = 0$.

c) $\det(A) \neq 0$.

d) The only solution is $x = A^{-1}b$.

e) $\mathrm{Nul}\,A = \{0\}$.

f) They are nonzero.

g) No, invertibility has nothing to do with diagonalizability.

5. Let A be an $n \times n$ matrix with characteristic polynomial $f(\lambda) = \det(A - \lambda I)$. (note: your instructor may have defined the characteristic polynomial as $\det(\lambda I - A)$. In either case, A will have the same eigenvalues and eigenvectors)

a) What is the degree of $f(\lambda)$?

b) Counting multiplicities, how many (real and complex) eigenvalues does A have?

c) If $f(0) = 0$, what does this tell you about A?

d) How can you know if A is diagonalizable?

e) If $n = 3$ and A has a complex eigenvalue, how many real roots does $f(\lambda)$ have?

f) Suppose $f(c) = 0$ for some real number c. How do you find the vectors x for which $Ax = cx$?

g) In general, do the roots of $f(\lambda)$ change when A is row reduced? Why or why not?

Solution.

a) n

b) n

c) A is not invertible, since 0 is an eigenvalue.

d) If f has n distinct roots, then A is diagonalizable. Otherwise, you have to check if the dimension of each eigenspace is equal to the algebraic multiplicity of the corresponding eigenvalue.

e) Complex roots come in pairs, so f has one real root.

f) You compute $\mathrm{Nul}(A - cI)$. (note: this is the same as $\mathrm{Nul}(cI - A)$)

g) Yes, row reduction does not preserve eigenvalues. For instance, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is row equivalent to $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$.

6. Find numbers a, b, c, and d such that the linear system corresponding to the augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & a \\ 0 & 4 & 5 & b \\ 0 & 0 & d & c \end{array}\right)$$

has **a)** no solutions, and **b)** infinitely many solutions.

Solution.

a) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ has no solutions.

b) $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 4 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$ has infinitely many solutions.

7. Celia has one hour to spend at the CRC, and she wants to jog, play handball, and ride a stationary bike. Jogging burns 13 calories per minute, handball burns 11, and cycling burns 7. She jogs twice as long as she rides the bike. How long should she participate in each of these activities in order to burn exactly 660 calories?

Solution.

Let x be the number of minutes spent jogging, y the number of minutes playing handball, and z the number of minutes cycling. The conditions of the problem require

$$\begin{aligned} x + y + z &= 60 \\ 13x + 11y + 7z &= 660 \\ x - 2z &= 0 \end{aligned} \qquad \iff \qquad \begin{pmatrix} 1 & 1 & 1 \\ 13 & 11 & 7 \\ 1 & 0 & -2 \end{pmatrix} x = \begin{pmatrix} 60 \\ 660 \\ 0 \end{pmatrix}.$$

We solve the matrix equation by using an augmented matrix and row reducing:

$$\begin{pmatrix} 1 & 1 & 1 & 60 \\ 13 & 11 & 7 & 660 \\ 1 & 0 & -2 & 0 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 3 & 60 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 60 \\ 0 \end{pmatrix} + z \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}.$$

So Celia should spend $2z$ minutes jogging, $60 - 3z$ minutes playing handball, and z minutes cycling, for any value of z strictly between 0 and 20 minutes (since she wants to do all three).

8. Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that rotates counterclockwise by $\dfrac{\pi}{6}$ radians, and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation that reflects about the line $y = x$.

a) Find the standard matrix A for T and the standard matrix B for U.

$$A = \begin{pmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

b) Find the matrix for T^{-1} and the matrix for U^{-1}. Clearly label your answers.

Recall the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

For T^{-1}: $A^{-1} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}$. For U^{-1}: $B^{-1} = \dfrac{1}{-1}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

(alternatively, A^{-1} is just *clockwise* rotation by $\pi/6$ radians)

c) Compute the matrix M for the linear transformation from $\mathbf{R}^2$ to $\mathbf{R}^2$ that first rotates *clockwise* by $\frac{\pi}{6}$ radians, then reflects about the line $y = x$, then rotates counterclockwise by $\frac{\pi}{6}$ radians.

This is the transformation that first does T^{-1}, then does U, then does T. In other words, we want the transformation for $(T \circ U \circ T^{-1})$.

$$\begin{aligned} M = ABA^{-1} &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix} \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix}. \end{aligned}$$

9. Let $W = \mathrm{Span}\left\{\begin{pmatrix}-6\\7\\2\end{pmatrix}, \begin{pmatrix}3\\2\\4\end{pmatrix}, \begin{pmatrix}4\\-1\\2\end{pmatrix}\right\}$. Find a basis for W and a basis for $W^\perp$.

Solution.

Let $W = \mathrm{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix}-6\\7\\2\end{pmatrix}, \qquad v_2 = \begin{pmatrix}3\\2\\4\end{pmatrix}, \qquad \text{and} \qquad v_3 = \begin{pmatrix}4\\-1\\2\end{pmatrix}.$$

First we compute a basis for W. Noting that W is the column space of

$$A = \begin{pmatrix}-6 & 3 & 4\\ 7 & 2 & -1\\ 2 & 4 & 2\end{pmatrix},$$

we row reduce to obtain

$$\begin{pmatrix}1 & 0 & -1/3\\ 0 & 1 & 2/3\\ 0 & 0 & 0\end{pmatrix}.$$

The first two columns are the pivot columns, so a basis of W is $\{v_1, v_2\}$. This means that W is the plane in $\mathbf{R}^3$ spanned by v_1 and v_2.

To get a basis for $W^\perp$, recall

$$W^\perp = \mathrm{Nul}(A^T) = \mathrm{Nul}\begin{pmatrix}-6 & 7 & 2\\ 3 & 2 & 4\\ 4 & -1 & 2\end{pmatrix}.$$

Row reducing A^T yields

$$\begin{pmatrix}1 & 0 & 8/11\\ 0 & 1 & 10/11\\ 0 & 0 & 0\end{pmatrix} \implies W^\perp = \mathrm{Nul}(A^T) = \mathrm{Span}\left\{\begin{pmatrix}-8\\-10\\11\end{pmatrix}\right\}.$$

10. Find a linear dependence relation among

$$v_1 = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 3 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ 5 \\ 3 \\ -1 \end{pmatrix}, \quad v_3 = \begin{pmatrix} 2 \\ -1 \\ 2 \\ 6 \end{pmatrix}, \quad v_4 = \begin{pmatrix} -1 \\ 4 \\ -5 \\ 1 \end{pmatrix}.$$

Which subsets of $\{v_1, v_2, v_3, v_4\}$ are linearly independent?

Solution.

A linear dependence relation is an equation of the form $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$, where c_1, c_2, c_3, c_4 are not all zero. This is the same as a nontrivial solution to the matrix equation

$$\begin{pmatrix} 1 & 1 & 2 & -1 \\ 4 & 5 & -1 & 4 \\ 0 & 3 & 2 & -5 \\ 3 & -1 & 6 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = 0.$$

Row reducing the matrix yields

$$\begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \implies \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{pmatrix} = c_4 \begin{pmatrix} -2 \\ 1 \\ 1 \\ 1 \end{pmatrix}.$$

Hence every linear dependence relation has the form $c_4(-2v_1 + v_2 + v_3 + v_4) = 0$. Taking $c_4 = 1$, a linear dependence relation is $-2v_1 + v_2 + v_3 + v_4 = 0$.

Any subset of size at most three vectors chosen from $\{v_1, v_2, v_3, v_4\}$ is linearly independent. If, for example, the set $\{v_1, v_2, v_3\}$ were linearly dependent, then there would exist a linear dependence relation $c_1v_1 + c_2v_2 + c_3v_3 = 0$. But this gives a linear dependence relation $c_1v_1 + c_2v_2 + c_3v_3 + 0v_4 = 0$, and we found above that no such relation exists (we found that all four coefficients must be nonzero in any linear dependence relation for these vectors).

11. Consider the matrix

$$A = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 8 & 4 \\ -1 & -4 & -2 \end{pmatrix}.$$

a) Find a basis for Col$A$.

b) Describe Col$A$ geometrically.

c) Find a basis for Nul$A$.

d) Describe Nul$A$ geometrically.

Solution.

First we row reduce A to get

$$\begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

a) The only pivot column is the first, so a basis for Col$A$ is

$$\left\{ \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \right\}.$$

b) This is the line through the first column of A.

c) The parametric vector form of $Ax = 0$ is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = y \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix},$$

which is the plane with basis

$$\left\{ \begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

d) As in Problem 6, we can compute that this is the plane in $\mathbf{R}^3$ defined by the equation $x + 4y + 2z = 0$.

12. Find the determinant of the matrix

$$A = \begin{pmatrix} 0 & 2 & -4 & 5 \\ 3 & 0 & -3 & 6 \\ 2 & 4 & 5 & 7 \\ 5 & -1 & -3 & 1 \end{pmatrix}.$$

Solution.

This is a big, complicated matrix. There is no convenient row or column for using the cofactor expansion, so it's probably best to find the determinant using row reduction. The number of computations required is probably beyond the scope of an online fill-in or multiple-choice exam. The answer is $\det(A) = 585$.

13. Let $A = \begin{pmatrix} 2 & -6 \\ 2 & 2 \end{pmatrix}$.

(a) Find the characteristic polynomial of A.
(b) Find the complex eigenvalues of A. Fully simplify your answer.
(c) For the eigenvalue with negative imaginary part, find a corresponding eigenvector.

Solution.

(a) The characteristic polynomial of A is given by

$$\det\begin{pmatrix} 2-\lambda & -6 \\ 2 & 2-\lambda \end{pmatrix} = (2-\lambda)(2-\lambda) + 12 = 4 - 4\lambda + \lambda^2 + 12 = \lambda^2 - 4\lambda + 16.$$

(b) $\lambda = \dfrac{4 \pm \sqrt{16-64}}{2} = \dfrac{4 \pm \sqrt{-48}}{2} = \dfrac{4 \pm 4\sqrt{3}i}{2} = 2 \pm 2\sqrt{3}i$

(c) For $\lambda = 2 - 2\sqrt{3}$, we have

$$\left(A - \lambda I \mid 0\right) = \left(\begin{array}{cc|c} 2-(2-2\sqrt{3}i) & -6 & 0 \\ (*) & (*) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 2\sqrt{3}i & -6 & 0 \\ (*) & (*) & 0 \end{array}\right)$$

so an eigenvector is $v = \begin{pmatrix} 6 \\ 2\sqrt{3}i \end{pmatrix}$. Other answers are possible.

For example, $v = \begin{pmatrix} -6 \\ -2\sqrt{3}i \end{pmatrix}$ is also an eigenvector, and so is $v = \begin{pmatrix} -i\sqrt{3} \\ 1 \end{pmatrix}$.

14. Find the eigenvalues and bases for the eigenspaces of the following matrices. Diagonalize if possible.

$$\textbf{a)}\ A = \begin{pmatrix} 4 & -3 & 3 \\ 0 & -2 & 4 \\ 0 & 0 & 2 \end{pmatrix} \qquad \textbf{b)}\ A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}.$$

Hint: the eigenvalues in (b) are $\lambda = -2$ and $\lambda = 4$.

Solution.

a) This is an upper-triangular matrix, so the eigenvalues are 4, -2, and 2. Computing the null spaces of $A-4I$, $A+2I$, and $A-2I$ yields bases

$$\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right\}, \qquad \text{and} \qquad \left\{\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\right\}$$

for the 4-, (-2)-, and 2-eigenspaces, respectively. Therefore,

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}^{-1}.$$

b) Computing the null spaces of $A+2I$ and $A-4I$ yields bases

$$\left\{\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}\right\} \qquad \text{and} \qquad \left\{\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}\right\}$$

of the (-2)- and 4-eigenspaces, respectively. Therefore,

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{pmatrix}^{-1}.$$

15. Find the least squares solution of the system of equations

$$\begin{aligned} x + 2y \phantom{{}+z} &= 0 \\ 2x + y + z &= 1 \\ 2y + z &= 3 \\ x + y + z &= 0 \\ 3x + 2z &= -1. \end{aligned}$$

Solution.

We have to find the least squares solution to $Ax = b$, where

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ 1 \\ 3 \\ 0 \\ -1 \end{pmatrix}.$$

We compute:

$$A^T A = \begin{pmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 15 & 5 & 9 \\ 5 & 10 & 4 \\ 9 & 4 & 7 \end{pmatrix}$$

$$A^T b = \begin{pmatrix} -1 \\ 7 \\ 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 15 & 5 & 9 & -1 \\ 5 & 10 & 4 & 7 \\ 9 & 4 & 7 & 2 \end{array}\right) \overset{\text{rref}}{\rightsquigarrow} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{187}{185} \\ 0 & 1 & 0 & \frac{137}{185} \\ 0 & 0 & 1 & \frac{43}{37} \end{array}\right).$$

Hence the least squares solution is

$$\widehat{x} = \left(-\frac{187}{185}, \frac{137}{185}, \frac{43}{37}\right).$$

16. Find A^{10} if $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$.

Solution.

This is a diagonalization problem. The characteristic polynomial is

$$f(\lambda) = -\lambda^3 + 5\lambda^2 - 8\lambda + 4.$$

We guess that f has an integer root, which then must divide 4. This works, and we factor

$$f(\lambda) = -(\lambda - 1)(\lambda - 2)^2.$$

We compute bases for the 1- and 2-eigenspaces, respectively, to be

$$\left\{ \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

It follows that

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1}$$

and therefore,

$$\begin{aligned} A^{10} &= \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1^{10} & 0 & 0 \\ 0 & 2^{10} & 0 \\ 0 & 0 & 2^{10} \end{pmatrix} \begin{pmatrix} -2 & -1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} -1022 & 0 & -2046 \\ 1023 & 1024 & 1023 \\ 1023 & 0 & 2047 \end{pmatrix}. \end{aligned}$$

17. Let $V = \mathrm{Span}\{v_1, v_2, v_3\}$, where

$$v_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \qquad v_2 = \begin{pmatrix}0\\1\\1\\1\end{pmatrix}, \qquad v_3 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}.$$

a) Find a basis for V.

b) Compute the matrix for the orthogonal projection onto V.

Solution.

a) Putting $v_1,\ v_2,\ v_3$ as columns of a matrix and row-reducing, we see they are linearly independent, so $\{v_1, v_2, v_3\}$ is a basis for V.

b) The matrix for projection onto V is

$$\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1/2 & 1/2\\ 0 & 0 & 1/2 & 1/2\end{pmatrix}.$$

18. Let W be the set of all vectors in $\mathbf{R}^3$ of the form $(x, x-y, y)$ where x and y are real numbers.

a) Find a basis for $W^\perp$.

b) Find the matrix B for orthogonal projection onto W.

c) Diagonalize B by finding an invertible matrix C and diagonal matrix D so that $B = CDC^{-1}$.

Solution.

a) A vector in W has the form

$$\begin{pmatrix}x\\x-y\\y\end{pmatrix} = \begin{pmatrix}x\\x\\0\end{pmatrix} + \begin{pmatrix}0\\-y\\y\end{pmatrix} = x\begin{pmatrix}1\\1\\0\end{pmatrix} + y\begin{pmatrix}0\\-1\\1\end{pmatrix}, \qquad \text{so } W \text{ has basis } \left\{\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\-1\\1\end{pmatrix}\right\}.$$

To get $W^\perp$ we find $\mathrm{Nul}\begin{pmatrix}1 & 1 & 0\\ 0 & -1 & 1\end{pmatrix}$ which gives us $x_1 = -x_3$, $x_2 = x_3$, and $x_3 = x_3$ (free), so $W^\perp$ has basis $\left\{\begin{pmatrix}-1\\1\\1\end{pmatrix}\right\}$.

b) Let A be the matrix whose columns are the basis vectors for W: $A = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}$. We calculate $A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, so

$$B = A(A^TA)^{-1}A^T = \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix}\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & -1 \\ 1 & -1 & 2 \end{pmatrix}.$$

c) The basis for W is a basis for the 1-eigenspace of B, while the basis for $W^\perp$ is a basis for the 0-eigenspace of B. Thus $B = CDC^{-1}$ where

$$C = \begin{pmatrix} 1 & 0 & -1 \\ 1 & -1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

19. Find, and draw, the best fit line $y = Mx + B$ through the points $(0,0)$, $(1,8)$, $(3,8)$, and $(4,20)$. Set up an equation (but do not solve) for the best-fit parabola $y = Ax^2 + Bx + C$ through those same data points.

Solution.

We want to find a least squares solution to the system of linear equations

$$\begin{aligned} 0 &= M(0) + B \\ 8 &= M(1) + B \\ 8 &= M(3) + B \\ 20 &= M(4) + B \end{aligned} \qquad \Longleftrightarrow \qquad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} M \\ B \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

Note the order of M (the slope) and B (the constant term) that we chose when forming the columns of our matrix A. This means that our least-squares answer will have first entry equal to the slope and second entry equal to the constant term of the best-fit line. We solve $A^TA\widehat{x} = A^Tb$ for $\widehat{x}$.

$$A^TA = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 1 \\ 4 & 1 \end{pmatrix} = \begin{pmatrix} 26 & 8 \\ 8 & 4 \end{pmatrix}$$

$$A^Tb = \begin{pmatrix} 0 & 1 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix} = \begin{pmatrix} 112 \\ 36 \end{pmatrix}$$

$$\left(A^TA \mid A^Tb\right) = \left(\begin{array}{cc|c} 26 & 8 & 112 \\ 8 & 4 & 36 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & 4 \\ 0 & 1 & 1 \end{array}\right).$$

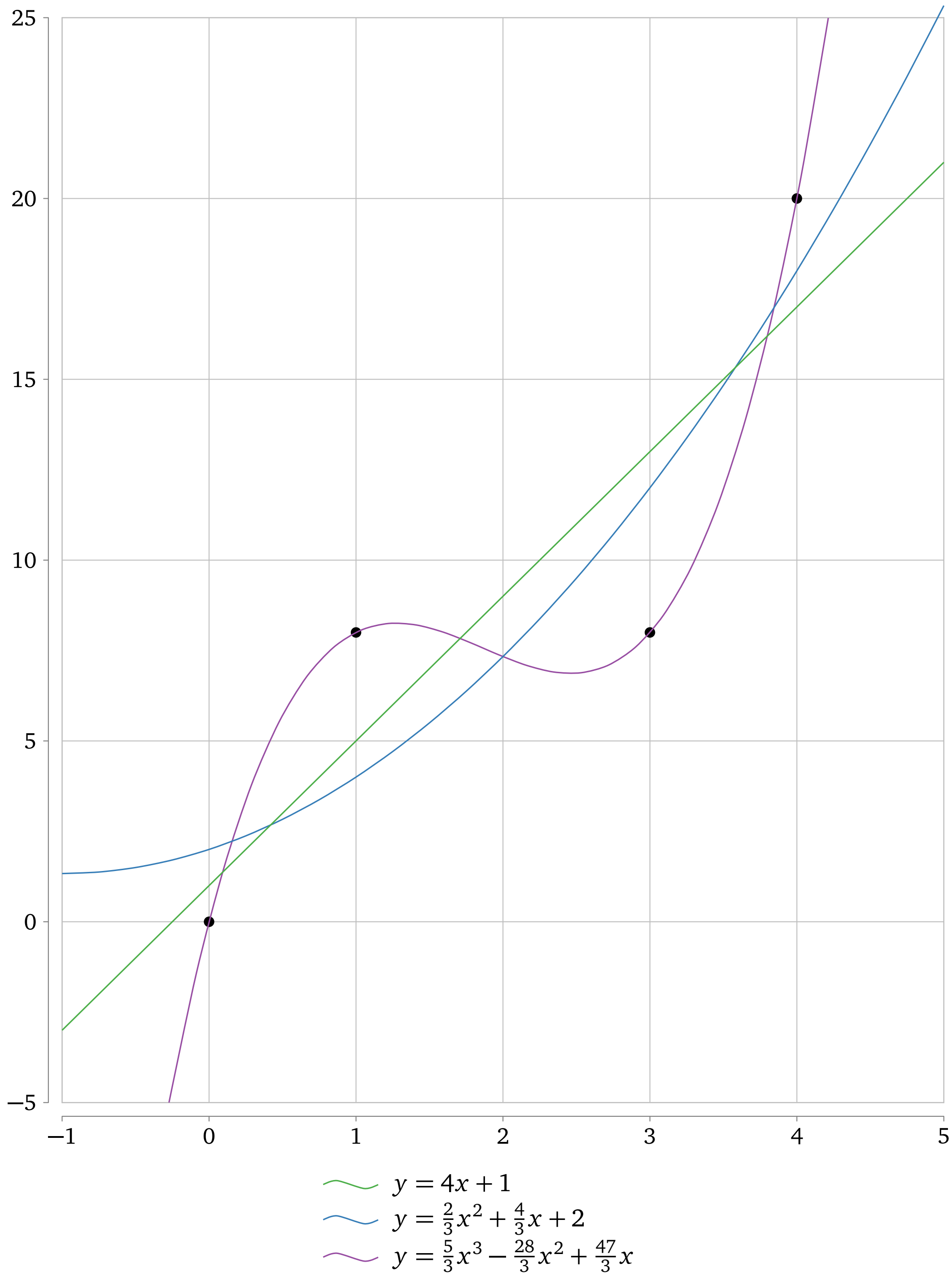
The least squares solution is $M = 4$ and $B = 1$, so the best fit line is $\boxed{y = 4x + 1}$.

Not all least-squares applications involve best-fit lines. Had we wanted a quadratic function to fit our data, we could have instead found the best-fit parabola $Ax^2 + Bx + C$. We would have gotten:

$$\begin{aligned} 0 &= A(0^2) + B(0) + C \\ 8 &= A(1^2) + B(1) + C \\ 8 &= A(3^2) + B(3) + C \\ 20 &= A(4^2) + B(4) + C \end{aligned} \quad \Longleftrightarrow \quad \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 9 & 3 & 1 \\ 16 & 4 & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \\ 8 \\ 20 \end{pmatrix}.$$

Painful computations would show that the least-squares solution is $A = 2/3$, $B = 4/3$, and $C = 2$, so the best fit quadratic is $y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$.

On the next page, you can find a picture with the best-fit line and best-fit parabola. The "best fit cubic" would be the cubic $y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$, which actually passes through all four data points.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5
0
−5
−1
0
1
2
3
4
5
$y = 4x + 1$
$y = \frac{2}{3}x^2 + \frac{4}{3}x + 2$
$y = \frac{5}{3}x^3 - \frac{28}{3}x^2 + \frac{47}{3}x$

Even more practice problems

Here is an additional list of practice problems.

Note: Answers to the remaining problems will **not** be posted.

0. Write down (and understand!!) the definitions of:

- Linear Dependence and Independence of Vectors
- Span of Vectors
- Echelon Form and Reduced Echelon Form
- Basis
- Subspace
- NullSpace of a Matrix
- Invertible and Non-invertible Matrix (Non-singular and singular matrix).
- Rank of a Matrix
- Column Space of a Matrix
- Row Space of a Matrix
- Dimension of Subspace
- Determinant of a Matrix
- Eigenvalue of a Matrix
- Eigenvector of a Matrix
- Characteristic polynomial of a Matrix
- Eigenspace corresponding to an Eigenvalue
- Algebraic and Geometric Multiplicity of an Eigenvalue
- Diagonalizable Matrix
- Dot Product
- Orthogonal Vectors
- Orthogonal Complement
- Orthogonal Projection
- Least Squares Solution

1. Suppose that A is a 3×3 matrix with eigenvalues 1, 2 and -5.

a) Find the determinant of A.

b) Is the matrix A invertible?

c) Is the matrix A diagonalizable?

d) Find the characteristic polynomial of A.

e) Find the eigenvalues of A^5.

f) Find the eigenvalues of BAB^{-1} where B is any invertible matrix.

2. V is a subspace of $\mathbf{R}^4$ spanned by the vectors $\begin{pmatrix} 2 \\ 3 \\ -1 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 3 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 7 \\ -1 \\ 2 \end{pmatrix}$. Let A be the 4×4 matrix with columns given by the 4 vectors above.

a) Find a basis of V, the dimension of V, the dimension of the kernel of A and the rank of A.

b) Is the matrix A invertible? Explain.

d) Is the vector $\begin{pmatrix} 0 \\ 1 \\ -3 \\ 4 \end{pmatrix}$ in V? If not, find the closest vector in V to it.

e) Find a basis of the null space of A.

f) Without performing further calculations, find determinant of A and explain your answer.

g) Find one eigenvalue and one eigenvector of A.

h) Repeat parts a)-g) with vectors $\begin{pmatrix} 1 \\ 3 \\ -2 \\ 5 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ 4 \\ -2 \\ 4 \end{pmatrix}$.

3. T is a linear transformation with $T\begin{pmatrix} 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}$ and $T\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

(a) Find $T\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

(b) Find a vector $\mathbf{u}$ such that $T(\mathbf{u}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

(c) Find the domain of T.

(d) Find the range of T.

(e) Find the matrix of T.

(f) Without doing further calculations, find the rank of the matrix of T and explain your answer.

(g) Does there exist a non-zero vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{0}$? Explain.

4. Consider the equation $A\mathbf{x} = \mathbf{b}$ with $A = \begin{pmatrix} 1 & -1 \\ 0 & -2 \\ 3 & 4 \\ 1 & 1 \end{pmatrix}$ and $b = \begin{pmatrix} -2 \\ 3 \\ 1 \\ 1 \end{pmatrix}$.

(a) Does this equation have a solution? If yes, find all solutions, if not, explain.

(b) Find the least squares solution of the above equation.

(c) Find the length and the dot product of the columns of A. Are the columns of A orthogonal?

(d) Find a pair of rows of A that are perpendicular to each other.

(e) Find the dimension of the kernel of A.

5. Find the eigenvalues, eigenvectors and diagonalize the following matrices, if possible.

(a) $\begin{pmatrix} 3 & -2 \\ -2 & 3 \end{pmatrix}$.

(b) $\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$.

(c) $\begin{pmatrix} 3 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

(d) $\begin{pmatrix} 5 & 3 & 1 \\ 0 & 5 & 1 \\ 1 & 0 & 2 \end{pmatrix}$.

6. Find the best fit line with equation $y = mx + b$ to the following sets of points:

(a) $(1,2)$, $(2,4)$, $(-1,0)$, $(5,2)$, $(3,3)$.

(b) $(2,-1)$, $(0,0)$, $(5,4)$, $(-1,2)$.

7. True or False. No partial credit.

(a) The span of the columns of a matrix A is equal to the range of the linear transformation T given by $T(\mathbf{x}) = A\mathbf{x}$.

(b) Any system of equations $A\mathbf{x} = \mathbf{b}$ has a least squares solution.

(c) Any 4 linearly independent vectors in $\mathbf{R}^4$ form a basis of $\mathbf{R}^4$.

(d) If the matrix A has more columns than rows then the system $A\mathbf{x} = \mathbf{0}$ always has infinitely many solutions.

(e) Any invertible matrix can be diagonalized.

(f) Any diagonalizable matrix is invertible.

(g) If $\mathbf{u}$ is perpendicular to every vector in the basis of a subspace V, then the orthogonal projection of $\mathbf{u}$ onto V is the zero vector.

(h) If the characteristic polynomial of A is $(\lambda-1)^2(\lambda-2)^2$ then the determinant of A is 2.

(i) For an invertible matrix A, the eigenvectors of A^{-1} are the same as eigenvectors of A.

(j) If a matrix A is not invertible then equation $A\mathbf{x}=\mathbf{b}$ has either no solutions of infinitely many solutions.

(k) If a matrix A is invertible then equation $A\mathbf{x}=\mathbf{b}$ always has a unique solution.

(l) If a $n\times n$ matrix A has linearly independent rows then A is invertible.

(m) If a $n\times n$ matrix A has linearly independent columns then A is invertible.

(n) If A is an invertible matrix then A^TA is also invertible.

(o) If 7×9 matrix A has kernel of dimension 5 then the column space of A has dimension 2.

(p) Any linearly independent set of vectors is a basis of its span.

(q) The eigenvalues of A are the same as eigenvalues of A^T.

(r) If a vector $\mathbf{u}$ is orthogonal to all rows of A then $\mathbf{u}$ is in the null space of A.

8. $A=\begin{pmatrix} 3 & s \\ 1 & -1 \end{pmatrix}$. Find a number s so that:

(a) A is non-invertible.

(b) A is not diagonalizable.

(c) 3 is an eigenvalue of A.

(d) Columns of A are orthogonal.

(e) $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ is an eigenvector of A.

(f) A^{-1} has eigenvalue 4.

(g) A^{-1} has eigenvector $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

9. $A = \begin{pmatrix} 3 & -3 & 0 \\ 3 & -1 & 2 \\ b & 0 & 2 \end{pmatrix}$. Find a number b (if possible) so that:

(a) The determinant of A is 4.

(b) The rank of A is 2.

(c) $\frac{1}{2}$ is an eigenvalue of A^{-1}.

(d) The system $A\mathbf{x} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ has no solutions.

(e) The system $A\mathbf{x} = \begin{pmatrix} -3 \\ 1 \\ 2 \end{pmatrix}$ has infinitely many solutions.

10. Find a 3×3 matrix with column space spanned by $\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$ and null space spanned by $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$.