Math 1553 Examination 3, SOLUTIONS, Fall 2024

Name		GT ID	

Circle your instructor and lecture below. Be sure to circle the correct choice!

Jankowski (A, 8:25-9:15)   Wessels(B, 8:25-9:15)   Hozumi (C, 9:30-10:20)

Wessels (D, 9:30-10:20)   Kim (G, 12:30-1:20)   Short (H, 12:30-1:20)

Shubin (I, 2:00-2:50)   He (L, 3:30-4:20)   Wan (M, 3:30-4:20)

Shubin (N, 5:00-5:50)   Denton (W, 8:25-9:15)

Please read the following instructions carefully.

- Write your initials at the top of each page. The maximum score on this exam is 70 points, and you have 75 minutes to complete it. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed. Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- On free response problems, show your work, unless instructed otherwise. A correct answer without appropriate work may receive little or no credit!
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.
- For questions with bubbles, either fill in the bubble completely or leave it blank. **Do not** mark any bubble with "X" or "/" or any such intermediate marking. Anything other than a blank or filled bubble may result in a 0 on the problem, and regrade requests may be rejected without consideration.

I affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until 7:45 PM on Wednesday, November 13.

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1. TRUE or FALSE. Clearly fill in the bubble for your answer. If the statement is *ever* false, fill in the bubble for False. You do not need to show any work, and there is no partial credit. Each question is worth 2 points.

(a) If A is a 3×3 matrix and $A \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, then A cannot be invertible.

● True

○ False

(b) If A and B are invertible $n \times n$ matrices, then $(AB)^{-1} = A^{-1}B^{-1}$.

○ True

● False

(c) Suppose A is a 3×3 matrix with characteristic polynomial

$$\det(A - \lambda I) = -\lambda^3 - 3\lambda^2 + 4.$$

Then the equation $Ax = 0$ must have only the trivial solution.

● True

○ False

(d) If A is an $n \times n$ matrix and $\det(A) = 3$, then $\lambda = 3$ must be an eigenvalue of A.

○ True

● False

(e) Suppose A is an $n \times n$ matrix and $\lambda = 2$ is an eigenvalue of A. If u and v are vectors in the 2-eigenspace of A, then $5u - 2v$ must be in the 2-eigenspace of A.

● True

○ False

Solution: Problem 1.

(a) True: by the Invertible Matrix Theorem, a square matrix A is invertible if and only if its corresponding transformation $T(x) = Ax$ is one-to-one. However, we have $Ar = As$ for the vectors $r = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$ and $s = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, so T is not one-to-one and therefore A is not invertible. Another way to see A is not invertible is to subtract the right equation from the left which gives

$$A \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} - A \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0, \qquad A \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} = 0,$$

so $Ax = 0$ has a non-trivial solution and therefore A is not invertible.

(b) False: the correct formula is $(AB)^{-1} = B^{-1}A^{-1}$.

(c) True: $\det(A) = \det(A - 0I) = 0 - 0 + 4$, so $\det(A) = 4$ therefore A is invertible. Therefore, the equation $Ax = 0$ has only the trivial solution.

(d) False: if $\det(A) = 3$ we cannot conclude that 3 is an eigenvalue. For example, the matrix $A = \begin{pmatrix} 3/2 & 0 \\ 0 & 2 \end{pmatrix}$ has determinant 3, however its eigenvalues are $\lambda_1 = 3/2$ and $\lambda_2 = 2$.

(e) True: the 2-eigenspace of A is a subspace of $\mathbf{R}^n$, so it is closed under addition and scalar multiplication. If u and v are in the 2-eigenspace, then so is every linear combination of u and v, therefore $5u - 2v$ is in the 2-eigenspace.

2. **Full solutions are on the next page.**

(a) (4 points) Suppose that A is an invertible $n \times n$ matrix. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

● Every vector in $\mathbf{R}^n$ is in the span of the columns of A.

● A does not have 0 as an eigenvalue.

● The RREF of A is the $n \times n$ identity matrix.

● The set $\{Ae_1, Ae_2, \ldots, Ae_n\}$ is a basis for $\mathbf{R}^n$.

(b) (2 points) Find the area of the parallelogram with vertices

$$(1,1), \quad (2,3), \quad (5,2), \quad (6,4).$$

Clearly fill in the bubble for the correct answer below.

○ 1 ○ 3 ● 7 ○ $\frac{7}{2}$ ○ $\frac{9}{2}$ ○ 9 ○ none of these

(c) (2 points) Suppose $\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 2$. Find $\det\begin{pmatrix} a & b & c \\ 4d-g & 4e-h & 4f-i \\ 3d & 3e & 3f \end{pmatrix}$.

Clearly fill in the bubble for the correct answer below.

○ 2 ○ −2 ○ 4 ○ −4 ● 6 ○ −6

○ 8 ○ −8 ○ 24 ○ −24 ○ none of these

(d) (2 points) Let $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$. Which **one** of the following is an eigenvector of A? Clearly fill in the bubble for your answer.

○ $\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$ ● $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

Solution:

(a) This is a classic Invertible Matrix Theorem problem.
(i) is true: the matrix transformation $T(x) = Ax$ is onto since A is invertible, therefore $\text{Col}(A) = \mathbf{R}^n$.

(ii) is true: $\lambda = 0$ is not an eigenvalue of A because $\det(A - 0I) = \det(A) \neq 0$.

(iii) is true: the $n \times n$ matrix A is invertible, therefore it has a pivot in every row and column, so the RREF of A is the identity matrix.

(iv) is true: the vectors in $\{Ae_1, Ae_2, \ldots, Ae_n\}$ are just the columns of A. Since A is invertible, the columns of A are linearly independent and span $\mathbf{R}^n$, therefore they form a basis for $\mathbf{R}^n$.

(b) We find two vectors that determine the parallelogram.

- The vector from $(1, 1)$ to $(2, 3)$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.
- The vector from $(1, 1)$ to $(5, 2)$ is $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$.

The area is $|\det(A)| = \left| \det \begin{pmatrix} 1 & 4 \\ 2 & 1 \end{pmatrix} \right| = |1(1) - 4(2)| = |-7| = 7.$

(c) The original determinant is 2, and we do row-operations.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \xrightarrow[\text{new det is -2}]{R_2 \leftrightarrow R_3} \begin{pmatrix} a & b & c \\ g & h & i \\ d & e & f \end{pmatrix} \xrightarrow[\text{new det is 2}]{R_2 = -R_2} \begin{pmatrix} a & b & c \\ -g & -h & -i \\ d & e & f \end{pmatrix}$$

$$\xrightarrow[\text{new det is 2}]{R_2 = R_2 + 4R_3} \begin{pmatrix} a & b & c \\ 4d - g & 4e - h & 4f - i \\ d & e & f \end{pmatrix} \xrightarrow[\text{new } \textbf{det is 6}]{R_3 = 3R_3} \begin{pmatrix} a & b & c \\ 4d - g & 4e - h & 4f - i \\ 3d & 3e & 3f \end{pmatrix}$$

(d) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ is never an eigenvector, $A \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 6 \\ 6 \end{pmatrix}$, and $A \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$.

However, $\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ is an eigenvector for $\lambda = 0$ since

$$A \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}.$$

3. Multiple choice and short answer. Parts (a)-(d) are unrelated and you do not need to show your work.

(a) (2 points) Find A^{-1} for the matrix $A = \begin{pmatrix} 5 & -3 \\ 2 & 1 \end{pmatrix}$.

Clearly fill in the bubble for your answer below.

○ $\frac{1}{11}\begin{pmatrix} 1 & -3 \\ 2 & 5 \end{pmatrix}$ ○ $\begin{pmatrix} -1 & 3 \\ -2 & -5 \end{pmatrix}$ ● $\frac{1}{11}\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}$ ○ $\begin{pmatrix} -1 & -3 \\ 2 & -5 \end{pmatrix}$

(b) (2 points) Let $A = \begin{pmatrix} -3 & -1 & 1 \\ 2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Find $\det(A^3)$. Clearly fill in the bubble for your answer below.

○ -2 ○ 2 ○ -6 ○ 6 ○ -8 ● 8

○ -1 ○ 1 ○ -16 ○ 16 ○ none of these

(c) (3 pts) Suppose A is a 2×2 matrix and the vectors $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ are eigenvectors of A. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

○ A is invertible.

○ $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ correspond to different eigenvalues of A.

● A is diagonalizable.

(d) (3 pts) Let $A = \begin{pmatrix} 1 & 5 & 0 \\ -3 & 7 & 0 \\ 0 & 2 & 1 \end{pmatrix} \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 5 & 0 \\ -3 & 7 & 0 \\ 0 & 2 & 1 \end{pmatrix}^{-1}$.

Which of the following statements must be true? Fill in the bubble for all that apply.

● $A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$.

○ $\det(A - I) = 0$.

● For each eigenvalue λ of A, the geometric multiplicity of λ is equal to the algebraic multiplicity of λ.

Solution:

(a) If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $ad - bc \neq 0$ then $A^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, so here

$$A^{-1} = \frac{1}{5(1) - 2(-3)}\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 1 & 3 \\ -2 & 5 \end{pmatrix}.$$

(b) Using the third row's cofactor expansion gives us

$$\det(A) = 1(-1)^{3+3}\det\begin{pmatrix} -3 & -1 \\ 2 & 0 \end{pmatrix} = 1(0+2) = 2,$$

so $\det(A^3) = (\det A)^3 = 2^3 = 8$.

(c) Statement (i) and (ii) are not necessarily true. For example, $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is not invertible but $A\begin{pmatrix} -1 \\ 2 \end{pmatrix} = 0\begin{pmatrix} -1 \\ 2 \end{pmatrix}$ and $A\begin{pmatrix} 4 \\ 3 \end{pmatrix} = 0\begin{pmatrix} 4 \\ 3 \end{pmatrix}$, so they are both eigenvectors of A for $\lambda = 0$.

Statement (iii) is true: the set $\left\{\begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 4 \\ 3 \end{pmatrix}\right\}$ is a basis for $\mathbf{R}^2$ consisting of eigenvectors of A, so A is diagonalizable by the Diagonalization Theorem.

(d) Statement (i) is true: from the diagonalization of A we see $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ is an eigenvector of A corresponding to the eigenvalue $\lambda = -1$, so

$$A\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

Statement (ii) is false: 1 is not an eigenvalue of A, so $\det(A - I) \neq 0$.

Statement (iii) is true. It is a consequence of the fact that A is diagonalizable, but we could also see it directly: the 7-eigenspace is 2-dimensional and the (-1)-eigenspace is 1-dimensional. The characteristic polynomial is degree 3 and the sum of the geo. mult. (each of which is no larger than the alg. mult.) is 3 by the previous sentence, so the geometric multiplicity of each eigenvalue must be equal to its algebraic multiplicity.

4. (a) (2 points) Find the value of a so that $a\cdot\det\begin{pmatrix} 10 & 20 & 8 \\ -2 & -4 & -1 \\ -4 & -1 & 6 \end{pmatrix} = \det\begin{pmatrix} 20 & 40 & 16 \\ 2 & 4 & 1 \\ -8 & -2 & 12 \end{pmatrix}$.

Clearly fill in the bubble for the correct answer below.

○ $a = 2$ ○ $a = -2$ ○ $a = 4$ ○ $a = 8$ ● $a = -4$

○ $a = -\frac{1}{8}$ ○ $a = \frac{1}{4}$ ○ $a = -\frac{1}{4}$ ○ none of these

(b) (4 points) Suppose A is a 3×3 matrix. Which of the following statements must be true? Clearly fill in the bubble for all that apply.

● If v_1, v_2, and v_3 are nonzero vectors in $\mathbf{R}^3$ satisfying $Av_1 = v_1$, $Av_2 = 3v_2$, and $Av_3 = -v_3$, then the set $\{v_1, v_2, v_3\}$ must be linearly independent.

○ The eigenvalues of A are its diagonal entries.

○ If $\det(A) = 3$, then $\det(2A^{-1}) = 2/3$.

● If A is invertible, then A^7 is invertible.

(c) (2 points) Suppose A is a 2×2 matrix satisfying

$$\text{Nul}(A - 3I) = \text{Span}\left\{\begin{pmatrix} 1 \\ -3 \end{pmatrix}\right\} \quad \text{and} \quad \text{Nul}(A + I) = \text{Span}\left\{\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right\}.$$

Find $A\begin{pmatrix} 0 \\ -4 \end{pmatrix}$. Hint: $\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Clearly fill in the bubble for the correct answer below.

○ $\begin{pmatrix} 0 \\ -4 \end{pmatrix}$ ○ $\begin{pmatrix} -2 \\ 10 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ○ $\begin{pmatrix} 2 \\ -10 \end{pmatrix}$ ○ $\begin{pmatrix} 0 \\ -2 \end{pmatrix}$

○ $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ ● $\begin{pmatrix} 4 \\ -8 \end{pmatrix}$ ○ $\begin{pmatrix} -4 \\ 10 \end{pmatrix}$ ○ none of these

(d) (2 points) Let A be the 2×2 matrix that reflects every vector $\begin{pmatrix} x \\ y \end{pmatrix}$ in $\mathbf{R}^2$ across the line $y = -3x$. Which **one** of the following vectors is in the (-1)-eigenspace of A? Clearly fill in the bubble for your answer.

○ $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$ ● $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -3 \end{pmatrix}$ ○ $\begin{pmatrix} -3 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Solution:

(a) To get from the first matrix to the second matrix, we multiply the first row by 2, then multiply the second row by -1, then multiply the third row by 2. This means we cumulatively multiply the first determinant by -4 to get to the second matrix, so $a = -4$.

(b) Statement (i) is true: eigenvectors corresponding to different eigenvalues are linearly independent.

Statement (ii) is often false when A is not triangular. For example, if A is the 3×3 matrix whose entries are all equal to 1, then its diagonal entries all 1, but it clearly has $\lambda = 0$ as an eigenvalue since it is not invertible.

Statement (iii) is false: $\det(A^{-1}) = \frac{1}{3}$, so $\det(2A^{-1}) = 2^3 \det(A^{-1}) = \frac{8}{3}$.

Statement (iv) is true: if A is invertible then $\det(A) \neq 0$, so $\det(A^7) = (\det\ A)^7 \neq 0$ which means A^7 is invertible.

(c) We're given $A\begin{pmatrix} 1 \\ -3 \end{pmatrix} = \begin{pmatrix} 3 \\ -9 \end{pmatrix}$ and $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$, so

$$A\begin{pmatrix} 0 \\ -4 \end{pmatrix} = A\left(\begin{pmatrix} 1 \\ -3 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 3 \\ -9 \end{pmatrix} - \begin{pmatrix} -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 4 \\ -8 \end{pmatrix}.$$

(d) The (-1)-eigenspace of A is the line perpendicular to the line of reflection $y = -3x$. This is the line through the origin with slope $\frac{1}{3}$, which is $y = x/3$, so we just need a vector on that line, which is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$.

Or: the line $y = x/3$ is $\text{Span}\left\{\begin{pmatrix} 1 \\ 1/3 \end{pmatrix}\right\}$ which is the same as $\text{Span}\left\{\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right\}$.

5. Short answer and free response. You do not need to show work on (a) and (b). On (c), show your work or you may receive little or no credit, even if your answer is correct.

(a) (2 points) Is there a diagonalizable 2×2 matrix A whose only eigenvalue is $\lambda = 3$? If your answer is yes, write such an A in the space below. If your answer is no, fill in the bubble labeled "no such A exists."

Yes, in fact $A = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$. This is the only example, since any such example would need to have two linearly independent eigenvectors so that $Av_1 = 3v_1$ and $Av_2 = 3v_2$, which leads to $Av = 3v$ for all v in $\mathbf{R}^2$, therefore $A = 3I$.

(b) (3 points) Is there a 3×3 triangular matrix B with exactly one eigenvalue $\lambda = 5$, whose 5-eigenspace is a plane? If your answer is yes, write such a B in the space below. If your answer is no, fill in the bubble labeled "no such B exists."

Yes: we put 5 on every diagonal entry, and then we make the entries either above or below the diagonal so that the system $(B - 5I)x = 0$ will have two free variables (so $B - 5I$ has one pivot). For example,

$$B = \begin{pmatrix} 5 & 1 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}.$$

(c) (5 points) Find A^{-1} for the matrix $A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}$. Enter your answer below.

$$\left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 2 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \xrightarrow{R_2 = R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{R_3 = R_3 - R_2} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 3 & -2 & 1 & 0 \\ 0 & 0 & -1 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow[\text{then } R_2 = R_2 - 3R_3]{R_3 = -R_3} \left(\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 4 & -2 & 3 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right)$$

$$\xrightarrow{R_1 = R_1 + R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & -2 & 3 \\ 0 & 1 & 0 & 4 & -2 & 3 \\ 0 & 0 & 1 & -2 & 1 & -1 \end{array}\right).$$

$$A^{-1} = \begin{pmatrix} 5 & -2 & 3 \\ 4 & -2 & 3 \\ -2 & 1 & -1 \end{pmatrix}$$

6. Free response. This page is worth 10 points. Show your work unless otherwise indicated! A correct answer without sufficient work may receive little or no credit.

Let $A = \begin{pmatrix} 5 & 2 \\ 4 & -2 \end{pmatrix}$.

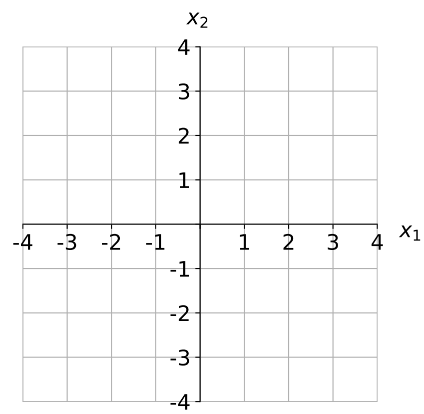
(a) Find the eigenvalues of A.

Solution: The characteristic equation is

$$0 = \det(A - \lambda I) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) = \lambda^2 - 3\lambda - 18 = (\lambda + 3)(\lambda - 6),$$

so $\lambda = -3$ and $\lambda = 6$.

(b) For each eigenvalue of A, find a basis for the corresponding eigenspace. Draw and label each eigenspace on the graph below.



Solution: $\left(A + 3I \mid 0 \right) = \left(\begin{array}{cc|c} 8 & 2 & 0 \\ 4 & 1 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 1/4 & 0 \\ 0 & 0 & 0 \end{array}\right)$, so the (-3)-eigenspace is given by the line $x_1 = -\frac{x_2}{4}$ where x_2 is free.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} -1/4 \\ 1 \end{pmatrix}. \quad \text{Basis for (-3)-eigenspace} : \left\{ \begin{pmatrix} -1/4 \\ 1 \end{pmatrix} \right\} \text{ or } \left\{ \begin{pmatrix} -1 \\ 4 \end{pmatrix} \right\}.$$

$\left(A - 6I \mid 0 \right) = \left(\begin{array}{cc|c} -1 & 2 & 0 \\ 4 & -8 & 0 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right)$, so the 6-eigenspace is the line $x_1 = 2x_2$ with x_2 free.

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \text{Basis for 6-eigenspace} : \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

(c) A is diagonalizable. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$ and write them in the space provided below. You do not need to show your work on this part. Many answers possible, for example

$$C = \begin{pmatrix} -1/4 & 2 \\ 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} -3 & 0 \\ 0 & 6 \end{pmatrix} \quad \text{or} \quad C = \begin{pmatrix} 2 & -1/4 \\ 1 & 1 \end{pmatrix}, \; D = \begin{pmatrix} 6 & 0 \\ 0 & -3 \end{pmatrix}.$$

7. Free response. Show your work! A correct answer without sufficient work will receive little or no credit. Parts (a) and (b) are unrelated.

(a) (4 points) Find det(A) for $A = \begin{pmatrix} 1 & -3 & 2 & 2 \\ 0 & 0 & 2 & 0 \\ 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 4 \end{pmatrix}$.

Enter your answer here by filling in the blank: $\det(A) = \mathbf{-24}$.

Solution: We use the cofactor expansion along the second row:

$$\begin{aligned} \det(A) &= 2(-1)^{2+3} \det \begin{pmatrix} 1 & -3 & 2 \\ 1 & 0 & 2 \\ 0 & 1 & 4 \end{pmatrix} \\ &= -2\Big(1(0-2) - (-3)(4-0) + 2(1-0)\Big) \\ &= -2(-2+12+2) = -24. \end{aligned}$$

(b) (6 points) Find the complex eigenvalues of the matrix $A = \begin{pmatrix} 3 & -10 \\ 1 & 1 \end{pmatrix}$. For the eigenvalue with **negative** imaginary part, find a corresponding eigenvector v. Simplify your eigenvalues as much as possible!

The eigenvalues are: $2+3i$ and $2-3i$ $\qquad v = \begin{pmatrix} 10 \\ 1+3i \end{pmatrix}$.

Solution: We solve for λ in the characteristic equation:

$$0 = \det(A - \lambda I) = \lambda^2 - \text{Tr}(A)\lambda + \det(A) = \lambda^2 - 4\lambda + 13,$$

$$\lambda = \frac{4 \pm \sqrt{(-4)^2 - 4(13)}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i.$$

We find v by using the 2×2 eigenvector trick for $\lambda = 2 - 3i$, where $a = 1 + 3i$ and $b = -10$ as we see below:

$$\big(A - (2-3i) \mid 0\big) = \left(\begin{array}{cc|c} 3-(2-3i) & -10 & 0 \\ (*) & (*) & 0 \end{array}\right) = \left(\begin{array}{cc|c} 1+3i & -10 & 0 \\ (*) & (*) & 0 \end{array}\right).$$

An eigenvector is $v = \begin{pmatrix} 10 \\ 1+3i \end{pmatrix}$. Other answers are possible, such as $v = \begin{pmatrix} -10 \\ -1-3i \end{pmatrix}$, or $\begin{pmatrix} 1-3i \\ 1 \end{pmatrix}$ or $\begin{pmatrix} -1+3i \\ -1 \end{pmatrix}$.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.