# Section 4.2

## Cofactor Expansions

## Orientation

Last time: we learned...

- ...the definition of the determinant.
- ...to compute the determinant using row reduction.
- ...all sorts of magical properties of the determinant, like
  - $\det(AB) = \det(A)\det(B)$
  - the determinant computes volumes
  - nonzero determinants characterize invertability
  - etc.

Today: we will learn...

- Special formulas for $2 \times 2$ and $3 \times 3$ matrices.
- How to compute determinants using *cofactor expansions.*
- How to compute inverses using determinants.

We already have a formula in the $2 \times 2$ case:

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

# Determinants of $3 \times 3$ Matrices

Here's the formula:

$$\det \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ \quad - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} \end{array}$$

How on earth do you remember this? Draw a bigger matrix, repeating the first two columns to the right:

$$+ \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix} - \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{11} & a_{12} \\ a_{21} & a_{22} & a_{23} & a_{21} & a_{22} \\ a_{31} & a_{32} & a_{33} & a_{31} & a_{32} \end{vmatrix}$$

Then add the products of the downward diagonals, and subtract the product of the upward diagonals. For example,

$$\det \begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = \begin{vmatrix} 5 & 1 & 0 & 5 & 1 \\ -1 & 3 & 2 & -1 & 3 \\ 4 & 0 & -1 & 4 & 0 \end{vmatrix} = -15 + 8 + 0 - 0 - 0 - 1 = -8$$

# Cofactor Expansions

When $n \geq 4$, the determinant isn't just a sum of products of diagonals. The formula is *recursive*: you compute a larger determinant in terms of smaller ones.

First some notation. Let $A$ be an $n \times n$ matrix.

$$\begin{aligned} A_{ij} &= ij\text{th } \textbf{minor} \text{ of } A \\ &= (n-1) \times (n-1) \text{ matrix you get by deleting the } i\text{th row and } j\text{th column} \end{aligned}$$

$$\begin{aligned} C_{ij} &= (-1)^{i+j} \det A_{ij} \\ &= ij\text{th } \textbf{cofactor} \text{ of } A \end{aligned}$$

The signs of the cofactors follow a checkerboard pattern:

$$\begin{pmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{pmatrix} \qquad \pm \text{ in the } ij \text{ entry is the sign of } C_{ij}$$

**Theorem**

The determinant of an $n \times n$ matrix $A$ is

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

This formula is called **cofactor expansion along the first row**.

This is the beginning of the recursion.

$$\det(\, a_{11} \,) = a_{11}.$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

The minors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{22} \end{pmatrix} \qquad A_{12} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{21} \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{12} \end{pmatrix} \qquad A_{22} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a_{11} \end{pmatrix}$$

The cofactors are

$$C_{11} = +\det A_{11} = a_{22} \qquad C_{12} = -\det A_{12} = -a_{21}$$

$$C_{21} = -\det A_{21} = -a_{12} \qquad C_{22} = +\det A_{22} = a_{11}$$

The determinant is

$$\det A = a_{11}C_{11} + a_{12}C_{12} = a_{11}a_{22} - a_{12}a_{21}.$$

# Cofactor Expansions

## $3 \times 3$ Matrices

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

The top row minors and cofactors are:

$$A_{11} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \qquad C_{11} = +\det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix}$$

$$A_{12} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} \qquad C_{12} = -\det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix}$$

$$A_{13} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \qquad C_{13} = +\det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix}$$

The determinant is magically the same formula as before:

$$\begin{aligned} \det A &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= a_{11}\det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} - a_{12}\det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + a_{13}\det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \end{aligned}$$

# Cofactor Expansions

## Example

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} = 5\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & -3 & 2 \\ 4 & 0 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$+\,0\cdot\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix}$$

$$= 5\cdot\det\begin{pmatrix} 3 & 2 \\ 0 & -1 \end{pmatrix} - 1\cdot\det\begin{pmatrix} -1 & 2 \\ 4 & -1 \end{pmatrix} + 0\cdot\det\begin{pmatrix} -1 & 3 \\ 4 & 0 \end{pmatrix}$$

$$= 5\cdot(-3-0) - 1\cdot(1-8)$$

$$= -15 + 7 = -8$$

# $2n-1$ More Formulas for the Determinant

Recall: the formula

$$\det(A) = \sum_{j=1}^{n} a_{1j} C_{1j} = a_{11} C_{11} + a_{12} C_{12} + \cdots + a_{1n} C_{1n}.$$

is called **cofactor expansion along the first row.** Actually, you can expand cofactors along any row or column you like!

## Magical Theorem

The determinant of an $n \times n$ matrix $A$ is

$$\det A = \sum_{j=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } i$$

$$\det A = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for any fixed } j$$

These formulas are called **cofactor expansion along the $i$th row**, respectively, **$j$th column**.

In particular, you get the *same answer* whichever row or column you choose.

Try this with a row or a column with a lot of zeros.

$$A = \begin{pmatrix} 2 & 1 & \boxed{\color{red}{0}} \\ 1 & 1 & \color{red}{0} \\ 5 & 9 & 1 \end{pmatrix}$$

It looks easiest to expand along the third column:

$$\det A = 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} - 0 \cdot \det \begin{pmatrix} \text{don't} \\ \text{care} \end{pmatrix} + 1 \cdot \det \begin{pmatrix} 2 & 1 & \cancel{0} \\ 1 & 1 & \cancel{0} \\ \cancel{5} & \cancel{9} & \cancel{1} \end{pmatrix}$$

$$= \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 2 - 1 = 1$$

- In general, computing a determinant by cofactor expansion is slower than by row reduction.
- It makes sense to expand by cofactors if you have a row or column with a lot of zeros.
- Also if your matrix has unknowns in it, since those are hard to row reduce (you don't know where the pivots are).

You can also use more than one method; for example:

- Use cofactors on a $4 \times 4$ matrix but compute the minors using the $3 \times 3$ formula.
- Do row operations to produce a row/column with lots of zeros, then expand cofactors (but keep track of how you changed the determinant!).

Example:

$$\det\begin{pmatrix} 5 & 1 & 0 \\ -1 & 3 & 2 \\ 4 & 0 & -1 \end{pmatrix} \xlongequal{R_2 = R_2 + 2R_3} \det\begin{pmatrix} 5 & 1 & 0 \\ 7 & 3 & 0 \\ 4 & 0 & -1 \end{pmatrix}$$

$$\xlongequal{\text{3rd column}} (-1)\det\begin{pmatrix} 5 & 1 \\ 7 & 3 \end{pmatrix} = -8$$

# Poll

**Poll**

$$\det \begin{pmatrix} 0 & 7 & 2 & 9 & 8 \\ 1 & 3 & 2 & 7 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix} = ?$$

**A.** $-84$ **B.** $-28$ **C.** $-7$ **D.** $0$ **E.** $7$ **F.** $28$ **G.** $84$

Repeatedly expanding along the first row, you get:

$$\det \begin{pmatrix} 0 & 7 & 2 & 9 & 8 \\ 1 & 3 & 2 & 7 & 4 \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 5 \end{pmatrix} = -1 \cdot \det \begin{pmatrix} 7 & 2 & 9 & 8 \\ 0 & 0 & 0 & 3 \\ 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 5 \end{pmatrix}$$

$$= (-1) \cdot 7 \cdot \det \begin{pmatrix} 0 & 0 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 5 \end{pmatrix} = (-1) \cdot 7 \cdot 3 \cdot \det \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

$$= (-1) \cdot 7 \cdot 3 \cdot 2 \cdot 2 = -84.$$

For $2 \times 2$ matrices we had a nice formula for the inverse:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} \\ C_{12} & C_{22} \end{pmatrix}.$$

### Theorem

This last formula works for any $n \times n$ invertible matrix $A$:

(3, 1) entry

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix} = \frac{1}{\det A} \left( C_{ij} \right)^{T}$$

Note that the cofactors are "transposed": the $(i, j)$ entry of the matrix is $C_{ji}$.

The proof uses Cramer's rule.

Compute $A^{-1}$, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The minors are:

$$A_{11} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{12} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{13} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$
$$A_{21} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{22} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \qquad A_{23} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$
$$A_{31} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \qquad A_{32} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A_{33} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The cofactors are (don't forget to multiply by $(-1)^{i+j}$):

$$\begin{array}{lll} C_{11} = -1 & C_{12} = \phantom{-}1 & C_{13} = -1 \\ C_{21} = \phantom{-}1 & C_{22} = -1 & C_{23} = -1 \\ C_{31} = -1 & C_{32} = -1 & C_{33} = \phantom{-}1 \end{array}$$

The determinant is (expanding along the first row):

$$\det A = 1 \cdot C_{11} + 0 \cdot C_{12} + 1 \cdot C_{13} = -2$$

# A Formula for the Inverse

## Example, continued

Compute $A^{-1}$, where $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

The inverse is

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} \\ C_{12} & C_{22} & C_{32} \\ C_{13} & C_{23} & C_{33} \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix}.$$

Check:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \cdot -\frac{1}{2} \begin{pmatrix} -1 & 1 & -1 \\ 1 & -1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

# A Formula for the Inverse

## Why?

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} C_{11} & C_{21} & C_{31} & \cdots & C_{n1} \\ C_{12} & C_{22} & C_{32} & \cdots & C_{n2} \\ C_{13} & C_{23} & C_{33} & \cdots & C_{n3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_{1n} & C_{2n} & C_{3n} & \cdots & C_{nn} \end{pmatrix}$$

That was a lot of work! It's way easier to compute inverses by row reduction.

- The formula is good for error estimates: the only division is by the determinant, so if your determinant is tiny, your error bars are large.
- It's also useful if your matrix has unknowns in it.
- It's part of a larger picture in the theory.

# Summary

We have several ways to compute the determinant of a matrix.

- Special formulas for $2 \times 2$ and $3 \times 3$ matrices.

  These work great for small matrices.

- Cofactor expansion.

  This is perfect when there is a row or column with a lot of zeros, or if your matrix has unknowns in it.

- Row reduction.

  This is the way to go when you have a big matrix which doesn't have a row or column with a lot of zeros.

- Any combination of the above.

  Cofactor expansion is recursive, but you don't have to use cofactor expansion to compute the determinants of the minors! Or you can do row operations and then a cofactor expansion.