Supplemental problems: §3.5-3.6

1. **a)** Fill in: A and B are invertible $n\times n$ matrices, then the inverse of AB is ________.

b) If the columns of an $n \times n$ matrix Z are linearly independent, is Z necessarily invertible? Justify your answer.

c) If A and B are $n \times n$ matrices and $ABx = 0$ has a unique solution, does $Ax = 0$ necessarily have a unique solution? Justify your answer.

Solution.

a) $(AB)^{-1} = B^{-1}A^{-1}$.

b) Yes. The transformation $x \to Zx$ is one-to-one since the columns of Z are linearly independent. Thus Z has a pivot in all n columns, so Z has n pivots. Since Z also has n rows, this means that Z has a pivot in every row, so $x \to Zx$ is onto. Therefore, Z is invertible.

Alternatively, since Z is an $n \times n$ matrix whose columns are linearly independent, the Invertible Matrix Theorem says that Z is invertible.

c) Yes. Since AB is an $n \times n$ matrix and $ABx = 0$ has a unique solution, the Invertible Matrix Theorem says that AB is invertible. Note A is invertible and its inverse is $B(AB)^{-1}$, since these are square matrices and

$$A(B(AB)^{-1}) = AB(AB)^{-1} = I_n.$$

Since A is invertible, $Ax = 0$ has a unique solution by the Invertible Matrix Theorem.

2. Suppose A is an invertible matrix and

$$A^{-1}e_1 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}, \qquad A^{-1}e_2 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix}, \qquad A^{-1}e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Find A.

Solution.

The columns of A^{-1} are

$$\begin{pmatrix} A^{-1}e_1 & A^{-1}e_2 & A^{-1}e_3 \end{pmatrix} \quad \text{so} \quad A = \begin{pmatrix} 4 & 3 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

To get A we find $(A^{-1})^{-1}$. Row-reducing $\left(A^{-1} \mid I\right)$ eventually gives us

$$\begin{pmatrix} 1 & 0 & 0 & \frac{2}{5} & -\frac{3}{5} & 0 \\ 0 & 1 & 0 & -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{so} \qquad A = \begin{pmatrix} \frac{2}{5} & -\frac{3}{5} & 0 \\ -\frac{1}{5} & \frac{4}{5} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$