Supplemental problems: §2.5

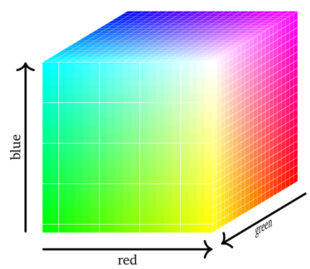
1. Justify why each of the following true statements can be checked without row reduction.

a) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ \pi \end{pmatrix}, \begin{pmatrix} 0 \\ \sqrt{2} \\ 0 \end{pmatrix} \right\}$ is linearly independent.

b) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix} \right\}$ is linearly independent.

c) $\left\{ \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 10 \\ 20 \end{pmatrix}, \begin{pmatrix} 0 \\ 5 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is linearly dependent.

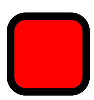
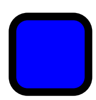
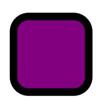
2. Every color on my computer monitor is a vector in $\mathbf{R}^3$ with coordinates between 0 and 255, inclusive. The coordinates correspond to the amount of red, green, and blue in the color.



Given colors $v_1, v_2, \ldots, v_p$, we can form a "weighted average" of these colors by making a linear combination

$$v = c_1 v_1 + c_2 v_2 + \cdots + c_p v_p$$

with $c_1 + c_2 + \cdots + c_p = 1$. Example:

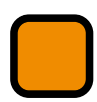
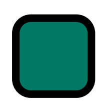
$\frac{1}{2}$  $+$ $\frac{1}{2}$  $=$ 

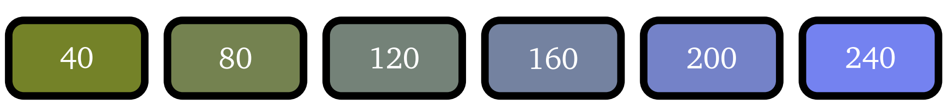
Consider the colors on the right. For which h is

$$\left\{ \begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}, \begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}, \begin{pmatrix} 116 \\ 130 \\ h \end{pmatrix} \right\}$$

linearly dependent? What does that say about the corresponding color?

$\begin{pmatrix} 180 \\ 50 \\ 200 \end{pmatrix}$ $\begin{pmatrix} 100 \\ 150 \\ 100 \end{pmatrix}$

$h =$ 

3. Which of the following must be true for any set of seven vectors in $\mathbf{R}^5$? Answer "yes", "no", or "maybe" in each case.

a) The vectors span $\mathbf{R}^5$.

b) The vectors are linearly dependent.

c) At least one of the vectors is in the span of the other six vectors.

d) If we put the seven vectors as the columns of a matrix A, then the matrix equation $Ax = 0$ must have infinitely many solutions.

e) Suppose we put the seven vectors as the columns of a matrix A. Then for each b in $\mathbf{R}^5$, the matrix equation $Ax = b$ must be consistent.

f) If every vector b in $\mathbf{R}^5$ can be written as a linear combination of our seven vectors, then in fact every b in $\mathbf{R}^5$ can be written in *infinitely many* different ways as a linear combination of our seven vectors.

4. Suppose A is a 2×3 matrix and the solution set to $Ax = 0$ is Span $\left\{ \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right\}$. Must it be true that the equation $Ax = b$ is consistent for each b in $\mathbf{R}^2$?

5. Write vectors u, v, and w in $\mathbf{R}^4$ so that $\{u, v, w\}$ is linearly dependent, but u is not in Span$\{v, w\}$.

6. Suppose that A is a matrix with columns v_1, v_2, v_3, and v_4, where

$$v_1 - 2v_2 + 3v_3 + v_4 = 0.$$

Consider the set of vectors $\{v_1, v_2, v_3, v_4\}$.

a) Write one nonzero vector so that $Ax = 0$.

b) Must it be true that every vector in the set $\{v_1, v_2, v_3, v_4\}$ is a linear combination of the other vectors in the set? Justify your answer.

Supplemental problems: §§2.6, 2.7, 2.9

1. Circle **TRUE** if the statement is always true, and circle **FALSE** otherwise.

a) If A is a 3×100 matrix of rank 2, then $\dim(\mathrm{Nul}A) = 97$.

TRUE **FALSE**

b) If A is an $m \times n$ matrix and $Ax = 0$ has only the trivial solution, then the columns of A form a basis for $\mathbf{R}^m$.

TRUE **FALSE**

c) The set $V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x - 4z = 0 \right\}$ is a subspace of $\mathbf{R}^4$.

TRUE **FALSE**

2. Write a matrix A so that $\mathrm{Col}A = \mathrm{Span}\left\{ \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \right\}$ and $\mathrm{Nul}A$ is the xz-plane.

3. Circle **T** if the statement is always true, and circle **F** otherwise. You do not need to explain your answer.

a) If $\{v_1, v_2, v_3, v_4\}$ is a basis for a subspace V of $\mathbf{R}^n$, then $\{v_1, v_2, v_3\}$ is a linearly independent set.

b) The solution set of a consistent matrix equation $Ax = b$ is a subspace.

c) A translate of a span is a subspace.

4. True or false (justify your answer). Answer true if the statement is *always* true. Otherwise, answer false.

a) There exists a 3×5 matrix with rank 4.

b) If A is an 9×4 matrix with a pivot in each column, then

$$\mathrm{Nul}A = \{0\}.$$

c) There exists a 4×7 matrix A such that nullity $A = 5$.

d) If $\{v_1, v_2, \ldots, v_n\}$ is a basis for $\mathbf{R}^4$, then $n = 4$.

5. Find bases for the column space and the null space of

$$A = \begin{pmatrix} 0 & 1 & -3 & 1 & 0 \\ 1 & -1 & 8 & -7 & 1 \\ -1 & -2 & 1 & 4 & -1 \end{pmatrix}.$$

6. Find a basis for the subspace V of $\mathbf{R}^4$ given by

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \text{ in } \mathbf{R}^4 \;\middle|\; x + 2y - 3z + w = 0 \right\}.$$

7. **a)** True or false: If A is an $m \times n$ matrix and $\mathrm{Nul}(A) = \mathbf{R}^n$, then $\mathrm{Col}(A) = \{0\}$.

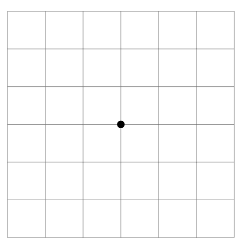
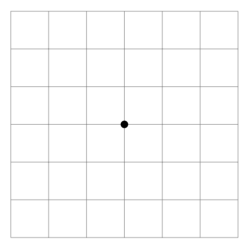
b) Give an example of 2×2 matrix whose column space is the same as its null space.

c) True or false: For some m, we can find an $m \times 10$ matrix A whose column span has dimension 4 and whose solution set for $Ax = 0$ has dimension 5.

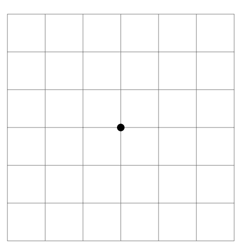
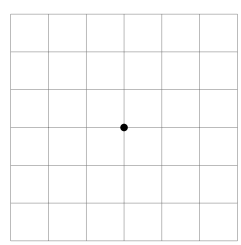
8. Suppose V is a 3-dimensional subspace of $\mathbf{R}^5$ containing $\begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Is $\left\{ \begin{pmatrix} 1 \\ -4 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 9 \\ 8 \\ 1 \\ 0 \\ 1 \end{pmatrix} \right\}$ a basis for V? Justify your answer.

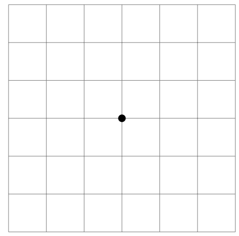
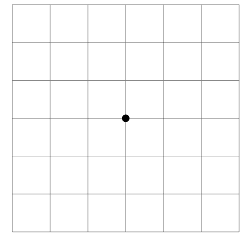
9. **a)** Write a 2×2 matrix A with **rank** 2, and draw pictures of $\mathrm{Nul}A$ and $\mathrm{Col}A$.

$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$ Nul A =  Col A = 

b) Write a 2×2 matrix B with **rank** 1, and draw pictures of $\mathrm{Nul}B$ and $\mathrm{Col}B$.

$B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$ Nul B =  Col B = 

c) Write a 2×2 matrix C with **rank** 0, and draw pictures of $\operatorname{Nul} C$ and $\operatorname{Col} C$.

$C = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$ Nul C =  Col C = 

(In the grids, the dot is the origin.)

Supplemental problems: §3.1

1. Review from 2.6-2.9. Fill in the blanks: If A is a 7×6 matrix and the solution set for $Ax = 0$ is a plane, then the column space of A is a ________-dimensional subspace of $\mathbf{R}^{\square}$.

2. Review from 2.6-2.9: Consider the matrix A below and its RREF:

$$A = \begin{pmatrix} 1 & 2 & -1 & -1 \\ -2 & -4 & -6 & 2 \\ 1 & 2 & -5 & -1 \end{pmatrix} \xrightarrow{RREF} \begin{pmatrix} 1 & 2 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

a) Write a basis for Col A.

b) Find a basis for Nul A.

c) Is there a matrix B so that $\mathrm{Col}(B) = \mathrm{Nul}(A)$? If yes, write such a B. If not, justify why no such matrix B exists.

3. Suppose T is a matrix transformation and the range of T is the subspace

$$V = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; x - 3y + 4z = 0 \right\}$$

of $\mathbf{R}^3$, which contains the vectors $v_1 = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}$. Is $\{v_1, v_2\}$ a basis for the range of T?

4. True or false. If the statement is *always* true, answer TRUE. Otherwise, circle FALSE.

a) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs reflection across the x-axis in $\mathbf{R}^2$. TRUE FALSE

b) The matrix transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ performs rotation counter-clockwise by $90°$ in $\mathbf{R}^2$. TRUE FALSE

5. Let T be the matrix transformation $T(x) = Ax$, where $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ -1 & 0 & -1 & -2 \\ 2 & 2 & 4 & 2 \end{pmatrix}$.
What is the domain of T? What is its codomain? Find a basis for the range of T and a basis for the kernel of T (the kernel of T is the set of all vectors satisfying $T(x) = 0$).

6. The matrix $A = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ has RREF $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}$. Define a matrix transformation by $T(x) = Ax$. Is $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$ a basis for the range of T?

7. In each case, a matrix is given below.

Match each matrix to its corresponding transformation (choosing from (i) through (viii)) by writing that roman numeral next to the matrix. Note there are four matrices and eight options, so not every option is used.

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$

(i) Reflection across x-axis
(ii) Reflection across y-axis
(iii) Scaling by a factor of 2
(iv) Scaling by a factor of 1/2
(v) Rotation counterclockwise by $\pi/2$ radians
(vi) Rotation clockwise by $\pi/2$ radians
(vii) Projection onto the x-axis
(viii) Projection onto the y-axis