Math 1553 Worksheet §3.2, 3.3

Solutions

1. Which of the following statements are true? Justify your answer.

a) Let A be a 3×3 matrix, such that $Ax = \begin{pmatrix} 1 \\ 5 \\ 2 \end{pmatrix}$ has a unique solution. Then, $Ax = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}$ also has a unique solution.

b) The transformation $T(x, y, z) = (y - 1, 4x + z, x)$ is an onto linear transformation.

c) Let A be a 3×4 matrix. Then, the transformation whose standard matrix is A cannot be onto.

Solution.

a) True. If $Ax = b$ has a unique solution (regardless of what b is), then so does $Ax = 0$, so the transformation whose standard matrix is A is one-to-one. Therefore, A has a pivot in each column (3 pivots). Since A is 3×3, this means A also has a pivot each row, so for every vector b' in $\mathbf{R}^3$, the system $Ax = b'$ is consistent and has unique solution.

b) False: T is not linear, since $T(0,0,0) = (-1,0,0)$. However, T is in fact onto. Given any vector (a, b, c) in $\mathbf{R}^3$, we could use the formula for T to find that there is a vector (x, y, z) so that $T(x, y, z) = (a, b, c)$, and in fact $x = c, y = a + 1, z = b - 4c$.

$$T(c,\ a+1,\ b-4c) = (a, b, c).$$

c) False. The matrix $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ is such matrix, since it has pivots in every row.

2. Which of the following transformations T are onto? Which are one-to-one? If the transformation is not onto, find a vector not in the range. If the transformation is not one-to-one, find two vectors with the same image.

a) Counterclockwise rotation by $32°$ in $\mathbf{R}^2$.

b) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (z, x)$.

c) The transformation $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (0, x)$.

d) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix}$.

e) The matrix transformation with standard matrix $A = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.

Solution.

a) This is both one-to-one and onto. If v is any vector in $\mathbf{R}^2$, then there is one and only one vector w such that $T(w) = v$, namely, the vector w that is rotated $32°$ *clockwise* from v.

b) This is onto. If (a, b) is any vector in the codomain $\mathbf{R}^2$, then $(a, b) = T(b, 0, a)$, so (a, b) is in the range. It is not one-to-one though: indeed, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$. Alternatively, we could have observed that T is a matrix transformation and examined its matrix A: $T(x) = Ax$ for

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since A has a pivot in every row but not every column, T is onto but not one-to-one.

c) This is not onto. There is no (x, y, z) such that $T(x, y, z) = (1, 0)$. It is not one-to-one: for instance, $T(0, 0, 0) = (0, 0) = T(0, 1, 0)$.

d) The transformation T with matrix A is onto if and only if A has a pivot in every *row*, and it is one-to-one if and only if A has a pivot in every *column*. So we row reduce:

$$A = \begin{pmatrix} 1 & 6 \\ -1 & 2 \\ 2 & -1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

This has a pivot in every column, so T is one-to-one. It does not have a pivot in every row, so it is not onto. To find a specific vector b in $\mathbf{R}^3$ which is not in the image of T, we have to find a $b = (b_1, b_2, b_3)$ such that the matrix equation $Ax = b$ is inconsistent. We row reduce again:

$$\left(A \mid b \right) = \left(\begin{array}{cc|c} 1 & 6 & b_1 \\ -1 & 2 & b_2 \\ 2 & -1 & b_3 \end{array}\right) \xrightarrow{\text{rref}} \left(\begin{array}{cc|c} 1 & 0 & \text{don't care} \\ 0 & 1 & \text{don't care} \\ 0 & 0 & -3b_1 + 13b_2 + 8b_3 \end{array}\right).$$

Hence any b_1, b_2, b_3 for which $-3b_1 + 13b_2 + 8b_3 \neq 0$ will make the equation $Ax = b$ inconsistent. For instance, $b = (1, 0, 0)$ is not in the range of T.

e) This matrix is already row reduced. We can see that does not have a pivot in every row *or* in every column, so it is neither onto nor one-to-one. In fact, if $T(x) = Ax$ then

$$T\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 3x_2 \\ x_3 \\ 0 \end{pmatrix},$$

so we can see that $(0, 0, 1)$ is not in the range of T, and that

$$T\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = T\begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}.$$

3. On your computer, go to the Interactive Transformation Challenge! Complete the Zoom, Reflect, and Scale challenges. If you complete a challenge in the optimal number of steps, the interactive demo will congratulate you. See if you can complete each of these challenges in the optimal number of steps.

4. The second little pig has decided to build his house out of sticks. His house is shaped like a pyramid with a triangular base that has vertices at the points $(0, 0, 0)$, $(2, 0, 0)$, $(0, 2, 0)$, and $(1, 1, 1)$.

The big bad wolf finds the pig's house and blows it down so that the house is rotated by an angle of 45° in a counterclockwise direction about the z-axis (look downward onto the xy- plane the way we usually picture the plane as $\mathbf{R}^2$), and then projected onto the xy-plane. Find the standard matrix A for the transformation T caused by the wolf.

Solution.

First notice that the shape of the pyramid has nothing to do with this problem. This is a question about the linear transformation T described in the last two lines.

To compute the matrix for T, we have to compute $T(e_1), T(e_2)$, and $T(e_3)$. To see the picture, let's put ourselves above the xy-plane (with the usual orientation of the x and y axes in the xy-plane), looking downward. For e_1 and e_2, it is as if we are applying $\begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, then putting a zero in the z-coordinate each time. We find

$$T(e_1) = T\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \qquad T(e_2) = T\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Rotating e_3 around the z-axis does nothing, and projecting onto the xy-plane sends it to zero, so $T(e_3) = 0$. Therefore, the matrix for T is

$$A = \begin{pmatrix} | & | & | \\ T(e_1) & T(e_2) & T(e_3) \\ | & | & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$