Math 1553 Worksheet §2.5, 2.6, 2.7, 2.9, 3.1 Solutions

- **1.** If the statement is always true, circle TRUE. Otherwise, circle FALSE. Justify your answer.
 - **a)** Suppose $A = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$ and $A \begin{pmatrix} -3 \\ 2 \\ 7 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$. Must v_1, v_2, v_3 be linearly

dependent? If yes, write a linear dependence relation for the vectors. **YES NO**

- **b)** If b is in Col(A), then so is 5b. **TRUE FALSE**
- c) In the following, A is an $m \times n$ matrix.
 - (1) **TRUE** FALSE If *A* has linearly dependent columns, then m < n.
 - (2) **TRUE** FALSE If *A* has linearly independent columns, then Ax = b must have at least one solution for each *b* in \mathbb{R}^m .
 - (3) **TRUE** FALSE If *b* is a vector in \mathbf{R}^m and Ax = b has exactly one solution, then $m \ge n$.

Solution.

- **a) YES**. By definition of matrix multiplication, $-3v_1 + 2v_2 + 7v_3 = 0$, so $\{v_1, v_2, v_3\}$ is linearly dependent and the equation gives a linear dependence relation.
- **b) TRUE**. Let *v* be a solution to Ax = b, so Av = b. Then A(5v) = 5Av = 5b.
- c) (1) FALSE For example $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

(Note that even though this part was false, there is a very similar-sounding statement that is true: if m < n, A must have linearly dependent columns.)

- (2) **FALSE** For example $A = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. There is no solution for Ax = b. (Note, however: if *A* has linearly independent columns, then the system Ax = 0 has no free variables, so Ax = b is either inconsistent or has a unique solution.)
- (3) **TRUE** If Ax = b has a unique solution, then since it is a translation of the solution set to Ax = 0, this means that Ax = 0 has only the trivial solution (no free variables). Thus, *A* has a pivot in every column, which is impossible if m < n (i.e. impossible if *A* has more columns than rows), so $m \ge n$.

a) If *A* is a 3×10 matrix with 2 pivots, then dim(Nul*A*) = 8 and rank(*A*) = 2.

TRUE FALSE

b) If A is an $m \times n$ matrix and Ax = 0 has only the trivial solution, then the transformation T(x) = Ax must have \mathbf{R}^m as its range.

TRUE FALSE

c) If {a, b, c} is a basis of a subspace V, then {a, a + b, b + c} is a basis of V as well.

TRUE FALSE

Solution.

- a) True. Recall that when we say a matrix has two pivots, we mean that its RREF has two pivots. rank(*A*) is the same as number of pivots in *A*. dim(Nul*A*) is the same as the number of free variables. Moreover by the Rank Theorem, rank(*A*) + dim(Nul*A*) = 10, so dim(Nul*A*) = 10 2 = 8.
- **b)** False. For example, $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ has only the trivial solution for Ax = 0, but

its column space is a 2-dimensional subspace of \mathbf{R}^3 .

c) True. Because *a* and *b* are independent, a + b and *a* are linearly independent, and furthermore *a* and *b* are in Span{a, a + b}. Next, *c* is independent from {a, b}, so b + c is independent from {a, a + b}, meaning that {a, a + b, b + c} is independent by the increasing span criterion. Since a, a + b, b + c are all clearly in Span{a, b, c}, by the basis theorem {a, a + b, b + c} also form a span for Span{a, b, c} = *V*. Alternatively, we could notice that *a*, *b*, and *c* are Span{a, a+b, b+c}, and since V = Span{a, b, c} it is a three-dimensional space spanned by the set of three elements {a, a+b, b+c}, those three elements must form a basis, by the basis theorem. **3.** Write a matrix *A* so that Col(*A*) is the solid blue line and Nul(*A*) is the dotted red line drawn below.



Solution.

We'd like to design an *A* with the prescribed column space $\operatorname{Span}\left\{\begin{pmatrix}1\\4\end{pmatrix}\right\}$ and null space $\operatorname{Span}\left\{\begin{pmatrix}3\\4\end{pmatrix}\right\}$

space Span $\left\{ \begin{pmatrix} 3 \\ -1 \end{pmatrix} \right\}$.

We start with analyzing the null space. We can write parametric form of the null space:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = t \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$
 is the same as $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -3x_2 \\ x_2 \end{pmatrix}$

Then this implies the RREF of *A* must be $\begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$.

Now we need to combine the information that column space is Span $\left\{ \begin{pmatrix} 1 \\ 4 \end{pmatrix} \right\}$. That means the second row must be 4 multiple of the first row. Therefore the second row must be (4 12). We conclude,

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 12 \end{pmatrix}$$

Note any nonzero scalar multiple of the above matrix is also a solution.

4. Let $A = \begin{pmatrix} 1 & -5 & -2 & -4 \\ 2 & 3 & 9 & 5 \\ 1 & 1 & 4 & 2 \end{pmatrix}$, and let *T* be the matrix transformation associated to *A*, so T(x) = Ax.

a) What is the domain of *T*? What is the codomain of *T*? Give an example of a vector in the range of *T*.

b) The RREF of A is
$$\begin{pmatrix} 1 & 0 & 3 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
.

(i) Write bases for Col(*A*) and Nul(*A*).

(ii) Is there a vector in the codomain of T which is not in the range of T? Justify your answer.

Solution.

a) The domain is \mathbf{R}^4 ; the codomain is \mathbf{R}^3 . The vector $\mathbf{0} = T(\mathbf{0})$ is contained in the range, as is

$$\begin{pmatrix} 1\\2\\1 \end{pmatrix} = T \begin{pmatrix} 1\\0\\0\\0 \end{pmatrix}.$$

b) (i) First, recall that the columns of *A*, which correspond to pivots in the RREF of *A*, form a basis for Col(*A*). We note that the first and second column of the RREF of *A* contain pivots. Therefore, the first and second columns of *A* form a basis for Col(*A*). That is,

$$\left\{ \begin{pmatrix} 1\\2\\1 \end{pmatrix}, \begin{pmatrix} -5\\3\\1 \end{pmatrix} \right\}.$$

Notice that the columns in the RREF of A do not form such basis themselves and in order to write a basis for Col(A), we need to use the corresponding columns in the matrix A itself.

In order to write a basis for Nul(A), we need to find the solution to the $\langle x \rangle$

matrix equation $A\begin{pmatrix} x_1\\ x_2\\ x_3\\ x_4 \end{pmatrix} = 0$ in parametric vector form. Since x_3 and x_4 are

,

free variables in this example. The parametric solution would be

$$\begin{cases} x_1 = -3x_3 - x_4 \\ x_2 = -x_3 - x_4 \end{cases}$$

which in vector form can be written as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = x_3 \begin{pmatrix} -3 \\ -1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \end{pmatrix}.$$

Therefore, a basis for Nul(*A*) is given by

$$\left\{ \begin{pmatrix} -3\\ -1\\ 1\\ 0 \end{pmatrix}, \begin{pmatrix} -1\\ -1\\ 0\\ 1 \end{pmatrix} \right\}.$$

(ii) Yes. The range of *T* is the column span of *A*, and *A* only has two pivots, so its column span is a 2-dimensional subspace of \mathbf{R}^3 . Since dim(\mathbf{R}^3) = 3, the range is not equal to \mathbf{R}^3 .