MATH 1553, SPRING 2023
PRACTICE FINAL

Name		**GT ID**	

Circle your lecture below.

Jankowski, lec. A and HP (8:25-9:15 AM) Jankowski, lecture D (9:30-10:20 AM)

Sane, lecture G (12:30-1:20 PM)

Sun, lecture I (2:00-2:50 PM) Sun, lecture M (3:30-4:20 PM)

Please **read all instructions** carefully before beginning.

- Write your initials at the top of each page.
- The maximum score on this exam is 100 points, and you have 170 minutes to complete this exam. Each problem is worth 10 points.
- There are no calculators or aids of any kind (notes, text, etc.) allowed.
- Simplify your answers as much as possible. For example, you may lose points if you do not simplify $\frac{8}{2}$ to 4, or if you do not simplify $\frac{0.1}{0.9}$ to $\frac{1}{9}$, etc.
- As always, RREF means "reduced row echelon form." The "zero vector" in $\mathbf{R}^n$ is the vector in $\mathbf{R}^n$ whose entries are all zero.
- Show your work, unless instructed otherwise. A correct answer without appropriate work will receive little or no credit!
- Unless stated otherwise, **the entries of all matrices on the exam are real numbers**.
- We will hand out loose scrap paper, but it **will not be graded** under any circumstances. All answers and all work must be written on the exam itself, with no exceptions.
- This exam is double-sided. You should have more than enough space to do every problem on the exam, but if you need extra space, you may use the *back side of the very last page of the exam*. If you do this, you must clearly indicate it.
- You may cite any theorem proved in class or in the sections we covered in the text.

Please read and sign the following statement.

I, the undersigned, hereby affirm that I will not share the contents of this exam with anyone. Furthermore, I have not received inappropriate assistance in the midst of nor prior to taking this exam. I will not discuss this exam with anyone in any form until after 9:00 PM on Tuesday, May 2.

Problem 1.

True or false. If the statement is *ever* false, circle FALSE. You do not need to show any work, and there is no partial credit. Each question is worth 1 point.

a) Let $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x + y + z = 0 \right\}$. Then W is a subspace of $\mathbf{R}^3$.

TRUE FALSE

b) If A is a matrix with more columns than rows, then the transformation $T(x) = Ax$ cannot be one-to-one.

TRUE FALSE

c) If A is a 6×11 matrix, then dim(Nul A) must be greater than dim(Col A).

TRUE FALSE

d) If A is a 3×3 matrix with characteristic polynomial $\det(A - \lambda I) = -(\lambda + 1)^2(\lambda - 2)$, then the 2-eigenspace of A is a line.

TRUE FALSE

e) If A is an $n \times n$ matrix and λ is an eigenvalue of A, then the columns of $A - \lambda I$ are linearly dependent.

TRUE FALSE

f) Let A be a 3×3 matrix. If dim(Nul A) $= 2$, then all of the eigenvalues of A must be real.

TRUE FALSE

g) Every inconsistent linear system of equations has exactly one least-squares solution.

TRUE FALSE

h) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$. Then $W^\perp = \text{Span}\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$.

TRUE FALSE

i) If A is a 5×5 matrix, then $(\text{Col } A)^\perp = \text{Nul } A$.

TRUE FALSE

j) Let W be a subspace of $\mathbf{R}^n$ and v be a vector in $\mathbf{R}^n$. If the orthogonal projection of v onto W is the zero vector, then v must be in $W^\perp$.

TRUE FALSE

Problem 2.

Multiple choice and short answer. You do not need to show your work, and there is no partial credit.

a) (3 points) Suppose $v_1,\ v_2,\ v_3$ are vectors in $\mathbf{R}^5$. Which of the following statements guarantee that $\{v_1, v_2, v_3\}$ is linearly independent? Circle *all* that apply.

(i) $\dim\big(\mathrm{Span}\{v_1, v_2, v_3\}\big) = 3$.

(ii) For some b in $\mathbf{R}^5$, the equation $x_1v_1 + x_2v_2 + x_3v_3 = b$ has exactly one solution.

(iii) v_3 is not a linear combination of v_1 and v_2.

b) (3 points) Let A and B be invertible $n \times n$ matrices. Which of the following matrices must also be invertible? Circle *all* that apply.

(i) AB

(ii) $A + B$

(iii) $-3A$

c) (4 points) Let $V = \mathrm{Nul}\begin{pmatrix} 1 & 4 & 4 \end{pmatrix}$. Which of the following options are a basis of V? Circle *all* that apply.

(i) $\left\{\begin{pmatrix} 1 \\ 4 \\ 4 \end{pmatrix}\right\}$

(ii) $\left\{\begin{pmatrix} -4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}\right\}$

(iii) $\left\{\begin{pmatrix} 8 \\ -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}\right\}$

(iv) $\left\{\begin{pmatrix} -4 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ -1 \\ 0 \end{pmatrix}\right\}$

Problem 3.

Multiple choice and short answer. You do not need to show your work, and there is no partial credit.

a) (2 pts) Suppose A is a 40×50 matrix and the dimension of the null space of A is 12. Which *one* of the following describes the column space of A?

(i) $\mathrm{Col}(A)$ is a 38-dimensional subspace of $\mathbf{R}^{40}$.

(ii) $\mathrm{Col}(A)$ is a 28-dimensional subspace of $\mathbf{R}^{50}$.

(iii) $\mathrm{Col}(A)$ is a 38-dimensional subspace of $\mathbf{R}^{50}$.

(iv) $\mathrm{Col}(A)$ is a 28-dimensional subspace of $\mathbf{R}^{40}$.

b) (4 points) Let $A = \begin{pmatrix} 4 & 1 \\ 2 & 1 \end{pmatrix}$.

(i) Find A^{-1}. Clearly circle your answer below.

(I) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$ (II) $A^{-1} = \frac{1}{-2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}$ (III) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -2 & 4 \end{pmatrix}$

(IV) $A^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$ (V) $A^{-1} = \begin{pmatrix} 1 & -2 \\ -1 & 4 \end{pmatrix}$ (VI) $A^{-1} = \frac{1}{2}\begin{pmatrix} -1 & 2 \\ 1 & -4 \end{pmatrix}$

(ii) Find $\det(3A)$.

(I) 2 (II) 6 (III) 9 (IV) 18 (V) 36 (VI) none of these

c) (4 points) Let A be an $n \times n$ matrix and let T be the matrix transformation $T(x) = Ax$. Which of the following conditions *guarantee* that A is invertible? Select *all* that apply.

(i) For some b in $\mathbf{R}^n$, the equation $Ax = b$ has exactly one solution.

(ii) The matrix transformation $T(x) = Ax$ is onto.

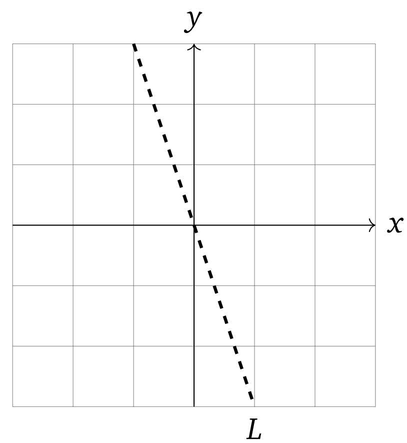
(iii) For each x in $\mathbf{R}^n$, there is a vector y in $\mathbf{R}^n$ so that $T(x) = y$.

(iv) A has exactly one eigenvalue $\lambda = 2$.

Problem 4.

Short answer and multiple choice. You do not need to show your work on this problem.

a) (4 points) Let $T : \mathbf{R}^2 \to \mathbf{R}^2$ be the linear transformation that reflects vectors across the **dashed** line L below, and let A be the standard matrix for T.



(i) Write all eigenvalues of A. ____________

(ii) For each eigenvalue of A, draw one eigenvector on the graph above. Clearly label the eigenvalue that corresponds to each eigenvector.

b) (4 points) Let A be a 4×3 matrix such that $\operatorname{rank}(A) = 1$. Let $T(v) = Av$ be the associated matrix transformation. For each question below, circle one answer.

(i) What is the domain of T? $\quad \mathbf{R} \quad \mathbf{R}^2 \quad$ a line in $\mathbf{R}^3 \quad \mathbf{R}^3 \quad \mathbf{R}^4$

(ii) What is the codomain of T? $\quad \mathbf{R} \quad \mathbf{R}^2 \quad$ a line in $\mathbf{R}^3 \quad \mathbf{R}^3 \quad \mathbf{R}^4$

(iii) What is the range of T, geometrically?
a point $\quad$ a line $\quad$ a plane $\quad$ 3-dimensional space

(iv) Fill in the blank: $\dim(\text{Row } A) =$ ____________.

c) (2 points) Suppose A is a 2×2 positive stochastic matrix with the property that as n gets very large, $A^n \begin{pmatrix} 200 \\ 200 \end{pmatrix}$ approaches $\begin{pmatrix} 300 \\ 100 \end{pmatrix}$. What is the steady-state vector w for A? Write w in the space below, and *simplify your answer*.

$$w = \begin{pmatrix} \\ \end{pmatrix}$$

Problem 5.

Short answer and multiple choice. You do not need to show your work on this problem, and there is no partial credit.

a) (2 points) Let A be a $n \times n$ matrix, and let u, v, w be nonzero vectors in $\mathbf{R}^n$ which are distinct (so $u \neq v$, $u \neq w$, and $v \neq w$). Suppose

$$Au = 2u, \quad Av = 2v, \quad Aw = -w.$$

Which one of the following vectors must be an eigenvector of A?

(i) $u - v$

(ii) $v - w$

(iii) $u - w$

(iv) none of the above

b) (2 points) Circle the matrix A below whose characteristic polynomial is

$$\det(A - \lambda I) = -\lambda^3 - \lambda.$$

(i) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (iii) $A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

(ii) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (iv) $A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$

c) (3 points) Suppose A is a 3×3 matrix with $\det(A) = -3$, and let T be the matrix transformation $T(x) = Ax$. which of the following must be true? Circle all that apply.

(i) $\det(2A) = -24$.

(ii) If S is a solid with volume 10, then the volume of $T(S)$ is 30.

(iii) $\lambda = -3$ is an eigenvalue of A.

d) (3 points) Suppose W is a subspace of $\mathbf{R}^n$ and P is the matrix for orthogonal projection onto W. Which of the following statements must be true? Circle *all* that apply.

(i) $P^4 = P$.

(ii) $I + P$ is invertible.

(iii) $I - P$ is invertible.

Problem 6.

Parts (a), (b), and (c) are unrelated and are 5 points, 2 points, and 3 points, respectively. You do not need to show your work on this page, and there is no partial credit.

a) Suppose W is a subspace of $\mathbf{R}^3$ and x is a vector in $\mathbf{R}^3$ whose orthogonal decomposition with respect to W is $x = x_W + x_{W^\perp}$, where

$$x_W = \begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix} \quad \text{and} \quad x_{W^\perp} = \begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}.$$

(i) What is the closest vector to x in W?

(I) $\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ (II) $\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$ (III) $\begin{pmatrix} 6 \\ 1 \\ 2 \end{pmatrix}$ (IV) $\begin{pmatrix} 4 \\ -5 \\ 0 \end{pmatrix}$

(V) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (VI) $\sqrt{30}\begin{pmatrix} 5 \\ -2 \\ 1 \end{pmatrix}$ (VII) $\sqrt{11}\begin{pmatrix} 1 \\ 3 \\ 1 \end{pmatrix}$

(ii) Find the distance from x to W.

(I) $\sqrt{30}$ (II) $\sqrt{11}$ (III) $\sqrt{41}$ (IV) $\sqrt{51}$

(V) 0 (VI) 30 (VII) 11 (VIII) 5

(iii) Is $\begin{pmatrix} 10 \\ -4 \\ 2 \end{pmatrix}$ in W? YES NO NOT ENOUGH INFORMATION

b) Let $u = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Let S be the set of all vectors $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbf{R}^2$ that satisfy $x \cdot u = 3$. Which one of the following describes S?

(i) S consists only of the vector $\begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix}$

(ii) S is the line $x_1 + 2x_2 = 3$ in $\mathbf{R}^2$.

(iii) $S = \text{Span}\left\{\begin{pmatrix} 3/5 \\ 6/5 \end{pmatrix}\right\}$

(iv) $S = \text{Span}\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

c) Let $W = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \text{ in } \mathbf{R}^3 \;\middle|\; x_1 - x_2 - 2x_3 = 0 \right\}$, and let B be the matrix for orthogonal projection onto W. Which of the following statements are true? Select *all* that apply.

(i) The eigenvalues of B are $\lambda = 0$ and $\lambda = 1$.

(ii) $B\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

(iii) The null space of B is 1-dimensional.

Problem 7.

Free response. Show your work in parts (a) and (e).

Let $T : \mathbf{R}^3 \to \mathbf{R}^2$ be the transformation such that $T\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y + 3z \\ z \end{pmatrix}$ and let $U : \mathbf{R}^2 \to \mathbf{R}^2$ be the transformation of rotation counterclockwise by 90 degrees.

a) (3 points) Write the standard matrix A for T. Enter your answer in the space below.

$$A = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

b) (2 points) Write the standard matrix B for U. Enter your answer in the space below.

$$B = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

c) (1 point) Is T onto? YES NO

d) (1 point) Circle the composition that makes sense: $T \circ U$ $U \circ T$.

e) (3 points) Using matrix multiplication, find the standard matrix C for the composition you chose above. Enter your answer in the space provided below.

$$C = \begin{pmatrix} \quad & \quad \\ \quad & \quad \end{pmatrix}$$

Problem 8.

Free response. Show your work! Parts (a) and (b) are unrelated.

a) (6 points) Let $A = \begin{pmatrix} 4 & 5 \\ -2 & -2 \end{pmatrix}$.

Find the eigenvalues of A. For the eigenvalue with *positive* imaginary part, find one corresponding eigenvector v. Enter your answers in the space provided below.

The eigenvalues are ____________________. [simplify the eigenvalues completely]

For the eigenvalue with *positive* imaginary part, an eigenvector is $v = \begin{pmatrix} \quad \end{pmatrix}$.

b) (4 pts) Let $W = \text{Span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$. Find the matrix B for orthogonal projection onto W.

Enter your answer in the space below.

$$B = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$

Problem 9.

Free response. Show your work! For this problem, let $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.

a) (3 points) Find all eigenvalues of A and write them in the box below.

b) (5 points) For each of the eigenvalues, find a basis of the corresponding eigenspace.

c) (2 points) A is diagonalizable. Write an invertible matrix C and a diagonal matrix D so that $A = CDC^{-1}$. Enter your answer below.

$C = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$ $\quad D = \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$

Problem 10.

Free response. Show your work.

Use least squares to find the best-fit line $y = Mx + B$ for the data points

$$(0, 11), \qquad (3, 2), \qquad (6, 5).$$

Enter your answer below:

$$y = \underline{\qquad\quad} x \ + \ \underline{\qquad\quad}.$$

You must show appropriate work and *simplify your answer completely* (if your answer has fractions, simplify them completely). If you simply guess a line or estimate the equation for the line based on the data points, you will receive little or no credit, even if your answer is correct or nearly correct.

This page is reserved ONLY for work that did not fit elsewhere on the exam.

If you use this page, please clearly indicate (on the problem's page and here) which problems you are doing.